China Team Selection Test 2003

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–	TST
Day 1	
1	ABC is an acute-angled triangle. Let D be the point on BC such that AD is the bisector of $\angle A$. Let E, F be the feet of perpendiculars from D to AC, AB respectively. Suppose the lines BE and CF meet at H. The circumcircle of triangle AFH meets BE at G (apart from H). Prove that the triangle constructed from BG, GE and BF is right-angled.
2	Suppose $A \subseteq \{0, 1, \dots, 29\}$. It satisfies that for any integer k and any two members $a, b \in A(a, b$ is allowed to be same), $a+b+30k$ is always not the product of two consecutive integers. Please find A with largest possible cardinality.
3	Suppose $A \subset \{(a_1, a_2, \dots, a_n) \mid a_i \in \mathbb{R}, i = 1, 2 \dots, n\}$. For any $\alpha = (a_1, a_2, \dots, a_n) \in A$ and $\beta = (b_1, b_2, \dots, b_n) \in A$, we define $$\gamma(\alpha, \beta) = (\lvert a_1 - b_1\rvert, \lvert a_2 - b_2\rvert, \dots, \lvert a_n - b_n\rvert),$$ $$D(A) = \{\gamma(\alpha, \beta) \mid \alpha, \beta \in A\}.$$ Please show that $\lvert D(A)\rvert \geq \lvert A\rvert$.
Day 2	
1	Find all functions $f : \mathbb{Z}^+ \to \mathbb{R}$, which satisfies $f(n+1) \geq f(n)$ for all $n \geq 1$ and $f(mn) = f(m)f(n)$ for all $(m, n) = 1$.
2	Suppose $A = \{1, 2, \dots, 2002\}$ and $M = \{1001, 2003, 3005\}$. B is an non-empty subset of A. B is called a M-free set if the sum of any two numbers in B does not belong to M. If $A = A_1 \cup A_2$, $A_1 \cap A_2 = \emptyset$ and A_1, A_2 are M-free sets, we call the ordered pair (A_1, A_2) a M-partition of A. Find the number of M-partitions of A.
3	Let (x_n) be a real sequence satisfying $x_0 = 0, x_2 = \sqrt[3]{2}x_1$, and $x_{n+1} = \frac{1}{\sqrt[3]{4}}x_n + \sqrt[3]{4}x_{n-1} + \frac{1}{2}x_{n-2}$ for every integer $n \geq 2$, and such that x_3 is a positive integer. Find the minimal number of integers belonging to this sequence.
–	Quiz 1

1 x, y and z are positive reals such that $x+y+z=xyz$. Find the minimum value of:

$$x^7(yz-1)+y^7(zx-1)+z^7(xy-1)$$

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2 In triangle ABC, the medians and bisectors corresponding to sides BC, CA, AB are m_a, m_b, m_c and w_a, w_b, w_c respectively. $P=w_a \cap m_b$, $Q=w_b \cap m_c$, $R=w_c \cap m_a$. Denote the areas of triangle ABC and PQR by F_1 and F_2 respectively. Find the least positive constant m such that $\frac{F_1}{F_2}<m$ holds for any $\triangle ABC$.

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3 There is a frog in every vertex of a regular 2n-gon with circumcircle($n \geq 2$). At certain time, all frogs jump to the neighborhood vertices simultaneously (There can be more than one frog in one vertex). We call it as *a way of jump*. It turns out that there is *a way of jump* with respect to 2n-gon, such that the line connecting any two distinct vertice having frogs on it after the jump, does not pass through the circumcentre of the 2n-gon. Find all possible values of n.

– Quiz 2

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1 Let $ABCD$ be a quadrilateral which has an incircle centered at O. Prove that

$$OA\cdot OC+OB\cdot OD=\sqrt{AB\cdot BC\cdot CD\cdot DA}$$

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2 Let $x<y$ be positive integers and $P=\frac{x^3-y}{1+xy}$. Find all integer values that P can take.

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3 The n roots of a complex coefficient polynomial $f(z)=z^n+a_1z^{n-1}+\cdots+a_{n-1}z+a_n$ are $z_1,z_2,\cdots,z_n$. If $\sum_{k=1}^n|a_k|^2\leq 1$, then prove that $\sum_{k=1}^n|z_k|^2\leq n$.

– Quiz 3

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1 m and n are positive integers. Set $A=\{1,2,\cdots,n\}$. Let set $B_n^m=\{(a_1,a_2\cdots,a_m)\mid a_i\in A, i=1,2,\cdots,m\}$ satisfying:
(1) $|a_i-a_{i+1}|\neq n-1$, $i=1,2,\cdots,m-1$; and

(2) at least three of $a_1,a_2,\cdots,a_m$ $(m\geq 3)$ are pairwise distince.

Find $|B_n^m|$ and $|B_6^3|$.

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2 Can we find positive reals $a_1,a_2,\dots,a_{2002}$ such that for any positive integer k, with $1\leq k\leq 2002$, every complex root z of the following polynomial $f(x)$ satisfies the condition $|\text{Im } z|\leq|\text{Re } z|$,

$$f(x)=a_{k+2001}x^{2001}+a_{k+2000}x^{2000}+\cdots+a_{k+1}x+a_k,$$

where $a_{2002+i} = a_i$, for $i = 1, 2, \ldots, 2001$.

3 Let $x_0 + \sqrt{2003}y_0$ be the minimum positive integer root of Pell function $x^2 - 2003y^2 = 1$. Find all the positive integer solutions (x, y) of the equation, such that x_0 is divisible by any prime factor of x.

– Quiz 4

1 In triangle ABC, $AB > BC > CA$, $AB = 6$, $\angle B - \angle C = 90^o$. The incircle touches BC at E and EF is a diameter of the incircle. Radical AF intersect BC at D. DE equals to the circumradius of $\triangle ABC$. Find BC and AC.

2 Find all functions f, g:$R \to R$ such that $f(x + yg(x)) = g(x) + xf(y)$ for $x, y \in R$.

3 Let $A = \{a_1, a_2, \cdots, a_n\}$ and $B = \{b_1, b_2 \cdots, b_n\}$ be two positive integer sets and $|A \cap B| = 1$. $C = \{$all the 2-element subsets of A$\} \cup \{$all the 2-element subsets of B$\}$. Function $f : A \cup B \to \{0, 1, 2, \cdots 2C_n^2\}$ is injective. For any $\{x, y\} \in C$, denote $|f(x) - f(y)|$ as the *mark* of $\{x, y\}$. If $n \geq 6$, prove that at least two elements in C have the same *mark*.

– Quiz 5

1 Let S be the set of points inside and on the boarder of a regular haxagon with side length 1. Find the least constant r, such that there exists one way to colour all the points in S with three colous so that the distance between any two points with same colour is less than r.

2 Denote by (ABC) the circumcircle of a triangle ABC.
Let ABC be an isosceles right-angled triangle with $AB = AC = 1$ and $\measuredangle CAB = 90^\circ$. Let D be the midpoint of the side BC, and let E and F be two points on the side BC.
Let M be the point of intersection of the circles (ADE) and (ABF) (apart from A).
Let N be the point of intersection of the line AF and the circle (ACE) (apart from A).
Let P be the point of intersection of the line AD and the circle (AMN).
Find the length of AP.

3 Sequence $\{a_n\}$ satisfies: $a_1 = 3$, $a_2 = 7$, $a_n^2 + 5 = a_{n-1}a_{n+1}$, $n \geq 2$. If $a_n + (-1)^n$ is prime, prove that there exists a nonnegative integer m such that $n = 3^m$.

– Quiz 6

1 Let $g(x) = \sum_{k=1}^n a_k \cos kx$, $a_1, a_2, \cdots, a_n, x \in R$. If $g(x) \geq -1$ holds for every $x \in R$, prove that $\sum_{k=1}^n a_k \leq n$.

2 Positive integer n cannot be divided by 2 and 3, there are no nonnegative integers a and b such that $|2^a - 3^b| = n$. Find the minimum value of n.

3 (1) D is an arbitary point in $\triangle ABC$. Prove that:

$$\frac{BC}{\min AD, BD, CD} \geq \{ \begin{array}{c} 2\sin A, \ \angle A < 90^o \\ \\ 2, \ \angle A \geq 90^o \end{array}$$

(2)E is an arbitary point in convex quadrilateral $ABCD$. Denote k the ratio of the largest and least distances of any two points among A, B, C, D, E. Prove that $k \geq 2\sin 70^o$. Can equality be achieved?

– Quiz 7

1 There are $n(n \geq 3)$ circles in the plane, all with radius 1. In among any three circles, at least two have common point(s), then the total area covered by these n circles is less than 35.

2 Given an integer $a_1(a_1 \neq -1)$, find a real number sequence $\{a_n\}(a_i \neq 0, i = 1, 2, \cdots, 5)$ such that $x_1, x_2, \cdots, x_5$ and $y_1, y_2, \cdots, y_5$ satisfy $b_{i1}x_1 + b_{i2}x_2 + \cdots + b_{i5}x_5 = 2y_i$, $i = 1, 2, 3, 4, 5$, then $x_1y_1 + x_2y_2 + \cdots + x_5y_5 = 0$, where $b_{ij} = \prod_{1 \leq k \leq i}(1 + ja_k)$.

3 Given S be the finite lattice (with integer coordinate) set in the xy-plane. A is the subset of S with most elements such that the line connecting any two points in A is not parallel to x-axis or y-axis. B is the subset of integer with least elements such that for any $(x, y) \in S$, $x \in B$ or $y \in B$ holds. Prove that $|A| \geq |B|$.

– Quiz 8

1 Triangle ABC is inscribed in circle O. Tangent PD is drawn from A, D is on ray BC, P is on ray DA. Line PU ($U \in BD$) intersects circle O at Q, T, and intersect AB and AC at R and S respectively. Prove that if $QR = ST$, then $PQ = UT$.

2 Let S be a finite set. f is a function defined on the subset-group 2^S of set S. f is called *monotonic decreasing* if when $X \subseteq Y \subseteq S$, then $f(X) \geq f(Y)$ holds. Prove that: $f(X \cup Y) + f(X \cap Y) \leq f(X) + f(Y)$ for $X, Y \subseteq S$ if and only if $g(X) = f(X \cup \{a\}) - f(X)$ is a *monotonic decreasing* funnction on the subset-group $2^{S \setminus \{a\}}$ of set $S \setminus \{a\}$ for any $a \in S$.

3 Let $a_1, a_2, ..., a_n$ be positive real number $(n \geq 2)$,not all equal,such that $\sum_{k=1}^n a_k^{-2n} = 1$,prove that:
$\sum_{k=1}^n a_k^{2n} - n^2. \sum_{1 \leq i < j \leq n} (\frac{a_i}{a_j} - \frac{a_j}{a_i})^2 > n^2$