Cono Sur Olympiad 2017

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Day 1 August 17th

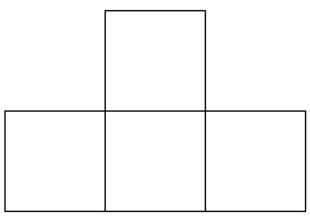
1 A positive integer n is called *guayaquilean* if the sum of the digits of n is equal to the sum of the digits of n^2. Find all the possible values that the sum of the digits of a guayaquilean number can take.

2 Let $A(XYZ)$ be the area of the triangle XYZ. A non-regular convex polygon $P_1P_2\dots P_n$ is called *guayaco* if exists a point O in its interior such that

$$A(P_1OP_2) = A(P_2OP_3) = \cdots = A(P_nOP_1).$$

Show that, for every integer $n \geq 3$, a guayaco polygon of n sides exists.

3 Let n be a positive integer. In how many ways can a $4 \times 4n$ grid be tiled with the following tetromino?



Day 2 August 18th

4 Let ABC an acute triangle with circumcenter O. Points X and Y are chosen such that:

-$\angle XAB = \angle YCB = 90^\circ$
-$\angle ABC = \angle BXA = \angle BYC$
-X and C are in different half-planes with respect to AB
-Y and A are in different half-planes with respect to BC

Prove that O is the midpoint of XY.

5 Let a, b and c positive integers. Three sequences are defined as follows:

- $a_1 = a$, $b_1 = b$, $c_1 = c$
- $a_{n+1} = \lfloor \sqrt{a_nb_n} \rfloor, \; b_{n+1} = \lfloor \sqrt{b_nc_n} \rfloor, \; c_{n+1} = \lfloor \sqrt{c_na_n} \rfloor$ for $n \geq 1$

-Prove that for any a, b, c, there exists a positive integer N such that $a_N = b_N = c_N$.
-Find the smallest N such that $a_N = b_N = c_N$ for some choice of a, b, c such that $a \geq 2$ y $b + c = 2a - 1$.

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6 The infinite sequence $a_1, a_2, a_3, \ldots$ of positive integers is defined as follows: $a_1 = 1$, and for each $n \geq 2$, a_n is the smallest positive integer, distinct from $a_1, a_2, \ldots, a_{n-1}$ such that:

$$\sqrt{a_n + \sqrt{a_{n-1} + \ldots + \sqrt{a_2 + \sqrt{a_1}}}}$$

is an integer. Prove that all positive integers appear on the sequence $a_1, a_2, a_3, \ldots$
