China Team Selection Test 2006

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by shobber, zhaobin

Day 1 March 19th

1 $ABCD$ is a trapezoid with $AB||CD$. There are two circles ω_1 and ω_2 is the trapezoid such that ω_1 is tangent to DA, AB, BC and ω_2 is tangent to BC, CD, DA. Let l_1 be a line passing through A and tangent to ω_2(other than AD), Let l_2 be a line passing through C and tangent to ω_1 (other than CB).

Prove that $l_1||l_2$.

2 Find all positive integer pairs (a,n) such that $\frac{(a+1)^n-a^n}{n}$ is an integer.

3 Given n real numbers a_1, a_2 $\dots$ a_n. ($n \geq 1$). Prove that there exists real numbers b_1, b_2 $\dots$ b_n satisfying:
(a) For any $1 \leq i \leq n$, $a_i - b_i$ is a positive integer.

(b)$\sum_{1 \leq i < j \leq n}(b_i - b_j)^2 \leq \frac{n^2-1}{12}$

Day 2 March 20th

1 Two positive valued sequences $\{a_n\}$ and $\{b_n\}$ satisfy:
(a): $a_0 = 1 \geq a_1$, $a_n(b_{n+1}+b_{n-1}) = a_{n-1}b_{n-1} + a_{n+1}b_{n+1}$, $n \geq 1$.

(b): $\sum_{i=1}^{n} b_i \leq n^{\frac{3}{2}}$, $n \geq 1$.

Find the general term of $\{a_n\}$.

2 Let ω be the circumcircle of $\triangle ABC$. P is an interior point of $\triangle ABC$. A_1, B_1, C_1 are the intersections of AP, BP, CP respectively and A_2, B_2, C_2 are the symmetrical points of A_1, B_1, C_1 with respect to the midpoints of side BC, CA, AB.
Show that the circumcircle of $\triangle A_2B_2C_2$ passes through the orthocentre of $\triangle ABC$.

3 Let a_i and b_i ($i = 1, 2, \cdots, n$) be rational numbers such that for any real number x there is:

$$x^2 + x + 4 = \sum_{i=1}^{n}(a_i x + b)^2$$

Find the least possible value of n.

Day 3 March 22nd

1 The centre of the circumcircle of quadrilateral $ABCD$ is O and O is not on any of the sides of $ABCD$. $P = AC \cap BD$. The circumecentres of $\triangle OAB$, $\triangle OBC$, $\triangle OCD$ and $\triangle ODA$ are O_1, O_2, O_3 and O_4 respectively.

Prove that O_1O_3, O_2O_4 and OP are concurrent.

2 $x_1, x_2, \cdots, x_n$ are positive numbers such that $\sum_{i=1}^{n} x_i = 1$. Prove that

$$\left(\sum_{i=1}^{n}\sqrt{x_i}\right)\left(\sum_{i=1}^{n}\frac{1}{\sqrt{1+x_i}}\right) \leq \frac{n^2}{\sqrt{n+1}}$$

3 d and n are positive integers such that $d \mid n$. The n-number sets $(x_1, x_2, \cdots x_n)$ satisfy the following condition:
(1) $0 \leq x_1 \leq x_2 \leq \cdots \leq x_n \leq n$

(2) $d \mid (x_1 + x_2 + \cdots x_n)$

Prove that in all the n-number sets that meet the conditions, there are exactly half satisfy $x_n = n$.

Day 4 March 24th

1 Let K and M be points on the side AB of a triangle $\triangle ABC$, and let L and N be points on the side AC. The point K is between M and B, and the point L is between N and C. If $\frac{BK}{KM} = \frac{CL}{LN}$, then prove that the orthocentres of the triangles $\triangle ABC$, $\triangle AKL$ and $\triangle AMN$ lie on one line.

2 Given three positive real numbers x, y, z such that $x + y + z = 1$, prove that
$\frac{xy}{\sqrt{xy+yz}} + \frac{yz}{\sqrt{yz+zx}} + \frac{zx}{\sqrt{zx+xy}} \leq \frac{\sqrt{2}}{2}$.

3 Find all second degree polynomial $d(x) = x^2 + ax + b$ with integer coefficients, so that there exists an integer coefficient polynomial $p(x)$ and a non-zero integer coefficient polynomial $q(x)$ that satisfy:

$$(p(x))^2 - d(x)\,(q(x))^2 = 1, \quad \forall x \in \mathbb{R}.$$

Day 5 March 26th

1 Let A be a non-empty subset of the set of all positive integers N^*. If any sufficient big positive integer can be expressed as the sum of 2 elements in A(The two integers do not have to be different), then we call that A is a divalent radical. For $x \geq 1$, let $A(x)$ be the set of all elements in A that do not exceed x, prove that there exist a divalent radical A and a constant number C so that for every $x \geq 1$, there is always $|A(x)| \leq C\sqrt{x}$.

2 The function $f(n)$ satisfies $f(0) = 0$, $f(n) = n - f(f(n-1))$, $n = 1, 2, 3 \cdots$. Find all polynomials $g(x)$ with real coefficient such that

$$f(n) = [g(n)], \qquad n = 0, 1, 2 \cdots$$

Where $[g(n)]$ denote the greatest integer that does not exceed $g(n)$.

3 Given positive integers m and n so there is a chessboard with mn 1×1 grids. Colour the grids into red and blue (Grids that have a common side are not the same colour and the grid in the left corner at the bottom is red). Now the diagnol that goes from the left corner at the bottom to the top right corner is coloured into red and blue segments (Every segment has the same colour with the grid that contains it). Find the sum of the length of all the red segments.

Day 6 March 28th

1 Let the intersections of $\odot O_1$ and $\odot O_2$ be A and B. Point R is on arc AB of $\odot O_1$ and T is on arc AB on $\odot O_2$. AR and BR meet $\odot O_2$ at C and D; AT and BT meet $\odot O_1$ at Q and P. If PR and TD meet at E and QR and TC meet at F, then prove: $AE \cdot BT \cdot BR = BF \cdot AT \cdot AR$.

2 Prove that for any given positive integer m and n, there is always a positive integer k so that $2^k - m$ has at least n different prime divisors.

3 k and n are positive integers that are greater than 1. N is the set of positive integers. $A_1, A_2, \cdots A_k$ are pairwise not-intersecting subsets of N and $A_1 \cup A_2 \cup \cdots \cup A_k = N$.

Prove that for some $i \in \{1, 2, \cdots, k\}$, there exsits infinity many non-factorable n-th degree polynomials so that coefficients of one polynomial are pairwise distinct and all the coeficients are in A_i.

Day 7 March 31st

1 H is the orthocentre of $\triangle ABC$. D, E, F are on the circumcircle of $\triangle ABC$ such that $AD \parallel BE \parallel CF$. S, T, U are the semetrical points of D, E, F with respect to BC, CA, AB. Show that S, T, U, H lie on the same circle.

2 Given positive integer n, find the biggest real number C which satisfy the condition that if the sum of the reciprocals of a set of integers (They can be the same.) that are greater than 1 is less than C, then we can divide the set of numbers into no more than n groups so that the sum of reciprocals of every group is less than 1.

3 For a positive integer M, if there exist integers a, b, c and d so that:

$$M \leq a < b \leq c < d \leq M+49, \qquad ad=bc$$

then we call M a GOOD number, if not then M is BAD. Please find the greatest GOOD number and the smallest BAD number.

Day 8 April 1st

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1 Let k be an odd number that is greater than or equal to 3. Prove that there exists a k^{th}-degree integer-valued polynomial with non-integer-coefficients that has the following properties:
(1) $f(0)=0$ and $f(1)=1$; and.
(2) There exist infinitely many positive integers n so that if the following equation:

$$n = f(x_1) + \cdots + f(x_s),$$

has integer solutions $x_1, x_2, \dots, x_s$, then $s \geq 2^k - 1$.

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2 Given positive integers m, a, b, $(a,b)=1$. A is a non-empty subset of the set of all positive integers, so that for every positive integer n there is $an \in A$ and $bn \in A$. For all A that satisfy the above condition, find the minimum of the value of $|A \cap \{1,2,\cdots,m\}|$

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3 $\triangle ABC$ can cover a convex polygon M.Prove that there exsit a triangle which is congruent to $\triangle ABC$ such that it can also cover M and has one side line paralel to or superpose one side line of M.
