China Team Selection Test 2007

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– TST

Day 1

1 Points A and B lie on the circle with center O. Let point C lies outside the circle; let CS and CT be tangents to the circle. M be the midpoint of minor arc AB of (O). MS, MT intersect AB at points E, F respectively. The lines passing through E, F perpendicular to AB cut OS, OT at X and Y respectively.
A line passed through C intersect the circle (O) at P, Q (P lies on segment CQ). Let R be the intersection of MP and AB, and let Z be the circumcentre of triangle PQR.
Prove that: X, Y, Z are collinear.

2 A rational number x is called *good* if it satisfies: $x = \frac{p}{q} > 1$ with p, q being positive integers, $\gcd(p, q) = 1$ and there exists constant numbers α, N such that for any integer $n \geq N$,

$$|\{x^n\} - \alpha| \leq \frac{1}{2(p+q)}$$

Find all the good numbers.

3 There are 63 points arbitrarily on the circle $\mathcal{C}$ with its diameter being 20. Let S denote the number of triangles whose vertices are three of the 63 points and the length of its sides is no less than 9. Fine the maximum of S.

Day 2

1 Find all functions $f : \mathbb{Q}^+ \mapsto \mathbb{Q}^+$ such that:

$$f(x) + f(y) + 2xyf(xy) = \frac{f(xy)}{f(x+y)}.$$

2 Let $x_1, \ldots, x_n$ be $n > 1$ real numbers satisfying $A = |\sum_{i=1}^n x_i| \neq 0$ and $B = \max_{1 \leq i < j \leq n} |x_j - x_i| \neq 0$. Prove that for any n vectors $\vec{\alpha_i}$ in the plane, there exists a permutation $(k_1, \ldots, k_n)$ of the numbers $(1, \ldots, n)$ such that

$$\left|\sum_{i=1}^n x_{k_i} \vec{\alpha_i}\right| \geq \frac{AB}{2A+B} \max_{1 \leq i \leq n} |\alpha_i|.$$

3 Let n be a positive integer, let A be a subset of $\{1, 2, \cdots, n\}$, satisfying for any two numbers $x, y \in A$, the least common multiple of x, y not more than n. Show that $|A| \leq 1.9\sqrt{n} + 5$.

– Quiz 1

1 When all vertex angles of a convex polygon are equal, call it equiangular. Prove that $p > 2$ is a prime number, if and only if the lengths of all sides of equiangular p polygon are rational numbers, it is a regular p polygon.

2 Let I be the incenter of triangle ABC. Let M, N be the midpoints of AB, AC, respectively. Points D, E lie on AB, AC respectively such that $BD = CE = BC$. The line perpendicular to IM through D intersects the line perpendicular to IN through E at P. Prove that $AP \perp BC$.

3 Prove that for any positive integer n, there exists only n degree polynomial $f(x)$, satisfying $f(0) = 1$ and $(x + 1)[f(x)]^2 - 1$ is an odd function.

– Quiz 2

1 $u, v, w > 0$,such that $u + v + w + \sqrt{uvw} = 4$
prove that $\sqrt{\frac{uv}{w}} + \sqrt{\frac{vw}{u}} + \sqrt{\frac{wu}{v}} \geq u + v + w$

2 Find all positive integers n such that there exists sequence consisting of 1 and $-1 : a_1, a_2, \cdots, a_n$ satisfying $a_1 \cdot 1^2 + a_2 \cdot 2^2 + \cdots + a_n \cdot n^2 = 0$.

3 Assume there are $n \geq 3$ points in the plane, Prove that there exist three points A, B, C satisfying $1 \leq \frac{AB}{AC} \leq \frac{n+1}{n-1}$.

– Quiz 3

1 Let ABC be a triangle. Circle ω passes through points B and C. Circle ω_1 is tangent internally to ω and also to sides AB and AC at T, P, and Q, respectively. Let M be midpoint of arc BC (containing T) of ω. Prove that lines PQ, BC, and MT are concurrent.

2 Given an integer $k > 1$. We call a $k-$digits decimal integer $a_1 a_2 \cdots a_k$ is $p-$monotonic, if for each of integers i satisfying $1 \leq i \leq k - 1$, when a_i is an odd number, $a_i > a_{i+1}$; when a_i is an even number, $a_i < a_{i+1}$. Find the number of $p-$monotonic $k-$digits integers.

3 Show that there exists a positive integer k such that $k \cdot 2^n + 1$ is composite for all $n \in \mathbb{N}_0$.

– Quiz 4

1 Let $a_1, a_2, \cdots, a_n$ be positive real numbers satisfying $a_1 + a_2 + \cdots + a_n = 1$. Prove that

$$(a_1a_2 + a_2a_3 + \cdots + a_na_1)\left(\frac{a_1}{a_2^2 + a_2} + \frac{a_2}{a_3^2 + a_3} + \cdots + \frac{a_n}{a_1^2 + a_1}\right) \geq \frac{n}{n+1}$$

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2 After multiplying out and simplifying polynomial $(x-1)(x^2-1)(x^3-1)\cdots(x^{2007}-1)$, getting rid of all terms whose powers are greater than 2007, we acquire a new polynomial $f(x)$. Find its degree and the coefficient of the term having the highest power. Find the degree of $f(x) = (1-x)(1-x^2)...(1-x^{2007}) \ (mod \ x^{2008})$.

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3 Let n be positive integer, $A, B \subseteq [0, n]$ are sets of integers satisfying $\mid A \mid + \mid B \mid \geq n + 2$. Prove that there exist $a \in A, b \in B$ such that $a + b$ is a power of 2.

– Quiz 5

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1 Let convex quadrilateral $ABCD$ be inscribed in a circle centers at O. The opposite sides BA, CD meet at H, the diagonals AC, BD meet at G. Let O_1, O_2 be the circumcenters of triangles AGD, BGC. O_1O_2 intersects OG at N. The line HG cuts the circumcircles of triangles AGD, BGC at P, Q, respectively. Denote by M the midpoint of PQ. Prove that $NO = NM$.

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2 Given n points arbitrarily in the plane $P_1, P_2, \ldots, P_n$, among them no three points are collinear. Each of P_i ($1 \leq i \leq n$) is colored red or blue arbitrarily. Let S be the set of triangles having $\{P_1, P_2, \ldots, P_n\}$ as vertices, and having the following property: for any two segments P_iP_j and P_uP_v, the number of triangles having P_iP_j as side and the number of triangles having P_uP_v as side are the same in S. Find the least n such that in S there exist two triangles, the vertices of each triangle having the same color.

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3 Find the smallest constant k such that
$\frac{x}{\sqrt{x+y}} + \frac{y}{\sqrt{y+z}} + \frac{z}{\sqrt{z+x}} \leq k\sqrt{x+y+z}$

for all positive x, y, z.

– Quiz 6

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1 Find all the pairs of positive integers (a, b) such that $a^2 + b - 1$ is a power of prime number $; a^2 + b + 1$ can divide $b^2 - a^3 - 1$, but it can't divide $(a + b - 1)^2$.

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2 Let $ABCD$ be the inscribed quadrilateral with the circumcircle ω.Let ζ be another circle that internally tangent to ω and to the lines BC and AD at points M, N respectively.Let I_1, I_2 be the incenters of the $\triangle ABC$ and $\triangle ABD$.Prove that M, I_1, I_2, N are collinear.

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3 Consider a 7×7 numbers table $a_{ij} = (i^2 + j)(i + j^2), 1 \le i, j \le 7$. When we add arbitrarily each term of an arithmetical progression consisting of 7 integers to corresponding to term of certain row (or column) in turn, call it an operation. Determine whether such that each row of numbers table is an arithmetical progression, after a finite number of operations.