China Team Selection Test 2008

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– TST

Day 1

1 Let ABC be a triangle, let $AB > AC$. Its incircle touches side BC at point E. Point D is the second intersection of the incircle with segment AE (different from E). Point F (different from E) is taken on segment AE such that $CE = CF$. The ray CF meets BD at point G. Show that $CF = FG$.

2 The sequence $\{x_n\}$ is defined by $x_1 = 2, x_2 = 12$, and $x_{n+2} = 6x_{n+1} - x_n$, $(n = 1, 2, \ldots)$. Let p be an odd prime number, let q be a prime divisor of x_p. Prove that if $q \neq 2, 3$, then $q \geq 2p - 1$.

3 Suppose that every positve integer has been given one of the colors red, blue,arbitrarily. Prove that there exists an infinite sequence of positive integers $a_1 < a_2 < a_3 < \cdots < a_n < \cdots$, such that inifinite sequence of positive integers $a_1, \frac{a_1+a_2}{2}, a_2, \frac{a_2+a_3}{2}, a_3, \frac{a_3+a_4}{2}, \cdots$ has the same color.

Day 2

4 Prove that for arbitary positive integer $n \geq 4$, there exists a permutation of the subsets that contain at least two elements of the set $G_n = \{1, 2, 3, \cdots, n\}$: $P_1, P_2, \cdots, P_{2^n-n-1}$ such that $|P_i \cap P_{i+1}| = 2, i = 1, 2, \cdots, 2^n - n - 2$.

5 For two given positive integers $m, n > 1$, let $a_{ij} (i = 1, 2, \cdots, n,\ j = 1, 2, \cdots, m)$ be nonnegative real numbers, not all zero, find the maximum and the minimum values of f, where

$$f = \frac{n\sum_{i=1}^{n}(\sum_{j=1}^{m} a_{ij})^2 + m\sum_{j=1}^{m}(\sum_{i=1}^{n} a_{ij})^2}{(\sum_{i=1}^{n}\sum_{j=1}^{m} a_{ij})^2 + mn\sum_{i=1}^{n}\sum_{j=1}^{m} a_{ij}^2}.$$

6 Find the maximal constant M, such that for arbitrary integer $n \geq 3$, there exist two sequences of positive real number $a_1, a_2, \cdots, a_n$, and $b_1, b_2, \cdots, b_n$, satisfying
(1):$\sum_{k=1}^{n} b_k = 1, 2b_k \geq b_{k-1} + b_{k+1}, k = 2, 3, \cdots, n-1;$
(2):$a_k^2 \leq 1 + \sum_{i=1}^{k} a_i b_i, k = 1, 2, 3, \cdots, n, a_n \equiv M.$

– Quiz 1

1 Let P be an arbitrary point inside triangle ABC, denote by A_1 (different from P) the second intersection of line AP with the circumcircle of triangle PBC and define B_1, C_1 similarly. Prove that $\left(1+2\cdot\frac{PA}{PA_1}\right)\left(1+2\cdot\frac{PB}{PB_1}\right)\left(1+2\cdot\frac{PC}{PC_1}\right)\geq 8$.

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2 Let $n>1$ be an integer, and n can divide $2^{\phi(n)}+3^{\phi(n)}+\cdots+n^{\phi(n)}$, let $p_1, p_2, \cdots, p_k$ be all distinct prime divisors of n. Show that $\frac{1}{p_1}+\frac{1}{p_2}+\cdots+\frac{1}{p_k}+\frac{1}{p_1p_2\cdots p_k}$ is an integer. (where $\phi(n)$ is defined as the number of positive integers $\leq n$ that are relatively prime to n.)

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3 Determine the greatest positive integer n such that in three-dimensional space, there exist n points $P_1, P_2, \cdots, P_n$, among n points no three points are collinear, and for arbitary $1\leq i<j<k\leq n$, $P_iP_jP_k$ isn't obtuse triangle.

– Quiz 2

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1 Let ABC be a triangle, line l cuts its sides BC, CA, AB at D, E, F, respectively. Denote by O_1, O_2, O_3 the circumcenters of triangle AEF, BFD, CDE, respectively. Prove that the orthocenter of triangle $O_1O_2O_3$ lies on line l.

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2 In a plane, there is an infinite triangular grid consists of equilateral triangles whose lengths of the sides are equal to 1, call the vertices of the triangles the lattice points, call two lattice points are adjacent if the distance between the two points is equal to 1;
A jump game is played by two frogs A, B, "A jump" is called if the frogs jump from the point which it is lying on to its adjacent point, " A round jump of A, B" is called if first A jumps and then B by the following rules:
Rule (1): A jumps once arbitrarily, then B jumps once in the same direction, or twice in the opposite direction;
Rule (2): when A, B sits on adjacent lattice points, they carry out Rule (1) finishing a round jump, or A jumps twice continually, keep adjacent with B every time, and B rests on previous position;
If the original positions of A, B are adjacent lattice points, determine whether for A and B,such that the one can exactly land on the original position of the other after a finite round jumps.

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3 Let z_1, z_2, z_3 be three complex numbers of moduli less than or equal to 1. w_1, w_2 are two roots of the equation $(z-z_1)(z-z_2)+(z-z_2)(z-z_3)+(z-z_3)(z-z_1)=0$. Prove that, for $j=1,2,3$, $\min\{|z_j-w_1|,|z_j-w_2|\}\leq 1$ holds.

– Quiz 3

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1 Let P be the the isogonal conjugate of Q with respect to triangle ABC, and P, Q are in the interior of triangle ABC. Denote by O_1, O_2, O_3 the circumcenters of triangle PBC, PCA, PAB, O'_1, O'_2, O'_3 the circumcenters of triangle QBC, QCA, QAB, O the circumcenter of triangle $O_1O_2O_3$, O' the circumcenter of triangle $O'_1O'_2O'_3$. Prove that OO' is parallel to PQ.

2 Prove that for arbitary integer $n > 16$, there exists the set S that contains n positive integers and has the following property:if the subset A of S satisfies for arbitary $a, a' \in A, a \neq a', a+a' \notin S$ holds, then $|A| \leq 4\sqrt{n}$.

3 Let $n > m > 1$ be odd integers, let $f(x) = x^n + x^m + x + 1$. Prove that $f(x)$ can't be expressed as the product of two polynomials having integer coefficients and positive degrees.

– Quiz 4

1 Given a rectangle $ABCD$, let $AB = b, AD = a(a \geq b)$, three points X, Y, Z are put inside or on the boundary of the rectangle, arbitrarily. Find the maximum of the minimum of the distances between any two points among the three points. (Denote it by a, b)

2 Let x, y, z be positive real numbers, show that $\frac{xy}{z} + \frac{yz}{x} + \frac{zx}{y} > 2\sqrt[3]{x^3 + y^3 + z^3}$.

3 Let S be a set that contains n elements. Let $A_1, A_2, \cdots, A_k$ be k distinct subsets of S, where $k \geq 2, |A_i| = a_i \geq 1 (1 \leq i \leq k)$. Prove that the number of subsets of S that don't contain any $A_i (1 \leq i \leq k)$ is greater than or equal to $2^n \prod_{i=1}^{k}(1 - \frac{1}{2^{a_i}})$.

– Quiz 5

1 Let ABC be an acute triangle, let M, N be the midpoints of minor arcs $\widehat{CA}, \widehat{AB}$ of the circumcircle of triangle ABC, point D is the midpoint of segment MN, point G lies on minor arc $\widehat{BC}$. Denote by I, I_1, I_2 the incenters of triangle ABC, ABG, ACG respectively.Let P be the second intersection of the circumcircle of triangle GI_1I_2 with the circumcircle of triangle ABC. Prove that three points D, I, P are collinear.

2 For a given integer $n \geq 2$, determine the necessary and sufficient conditions that real numbers $a_1, a_2, \cdots, a_n,$ not all zero satisfy such that there exist integers $0 < x_1 < x_2 < \cdots < x_n$, satisfying $a_1x_1 + a_2x_2 + \cdots + a_nx_n \geq 0$.

3 Let $0 < x_1 \leq \frac{x_2}{2} \leq \cdots \leq \frac{x_n}{n}, 0 < y_n \leq y_{n-1} \leq \cdots \leq y_1$, Prove that $(\sum_{k=1}^{n} x_ky_k)^2 \leq (\sum_{k=1}^{n} y_k)(\sum_{k=1}^{n}(x_k^2 - \frac{1}{4}x_kx_{k-1})y_k)$. where $x_0 = 0$.

– Quiz 6

1 Prove that in a plane, arbitrary n points can be overlapped by discs that the sum of all the diameters is less than n, and the distances between arbitrary two are greater than 1. (where the distances between two discs that have no common points are defined as that the distances between its centers subtract the sum of its radii; the distances between two discs that have common points are zero)

2 Prove that for all $n \geq 2$, there exists n-degree polynomial $f(x) = x^n + a_1 x^{n-1} + \cdots + a_n$ such that
(1) $a_1, a_2, \cdots, a_n$ all are unequal to 0;
(2) $f(x)$ can't be factorized into the product of two polynomials having integer coefficients and positive degrees;
(3) for any integers $x, |f(x)|$ isn't prime numbers.

3 Find all positive integers n having the following properties:in two-dimensional Cartesian coordinates, there exists a convex n lattice polygon whose lengths of all sides are odd numbers, and unequal to each other. (where lattice polygon is defined as polygon whose coordinates of all vertices are integers in Cartesian coordinates.)