**China Team Selection Test 2009**

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| – | TST |
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| 1 | Let $ABC$ be a triangle. Point $D$ lies on its sideline $BC$ such that $\angle CAD = \angle CBA$. Circle $(O)$ passing through $B, D$ intersects $AB, AD$ at $E, F$, respectively. $BF$ meets $DE$ at $G$.Denote by$M$ the midpoint of $AG$. Show that $CM\perp AO$. |
| 2 | Given an integer $n\ge 2$, find the maximal constant $\lambda(n)$ having the following property: if a sequence of real numbers $a_0,a_1,a_2,\cdots,a_n$ satisfies $0 = a_0\le a_1\le a_2\le\cdots\le a_n$, and $a_i\ge\frac{1}{2}(a_{i+1}+a_{i-1}), i=1,2,\cdots,n-1$, then $(\sum_{i=1}^n ia_i)^2\ge\lambda(n)\sum_{i=1}^n a_i^2$. |
| 3 | Prove that for any odd prime number $p$, the number of positive integer $n$ satisfying $p\mid n!+1$ is less than or equal to $cp^{\frac{2}{3}}$. where $c$ is a constant independent of $p$. |
| 4 | Let positive real numbers $a, b$ satisfy $b-a>2$. Prove that for any two distinct integers $m, n$ belonging to $[a, b)$, there always exists non-empty set $S$ consisting of certain integers belonging to $[ab, (a+1)(b+1))$ such that $\frac{\prod\limits_{x\in S}}{mn}$ is square of a rational number. |
| 5 | Let $m>1$ be an integer, $n$ is an odd number satisfying $3\le n<2m$, number $a_{i,j} (i,j\in N, 1\le i\le m, 1\le j\le n)$ satisfies $(1)$ for any $1\le j\le n, a_{1,j}, a_{2,j},\cdots,a_{m,j}$ is a permutation of $1,2,3,\cdots,m$; $(2)$ for any $1<i\le m, 1\le j\le n-1, \vert a_{i,j}-a_{i,j+1}\vert\le 1$ holds. Find the minimal value of $M$, where $M=max_{1<i<m}\sum_{j=1}^n a_{i,j}$. |
| 6 | Determine whether there exists an arithimethical progression consisting of 40 terms and each of whose terms can be written in the form $2^m+3^n$ or not. where $m, n$ are nonnegative integers. |
| – | Quiz 1 |
| 1 | Given that circle $\omega$ is tangent internally to circle $\Gamma$ at $S$. $\omega$ touches the chord $AB$ of $\Gamma$ at $T$. Let $O$ be the center of $\omega$. Point $P$ lies on the line $AO$. Show that $PB\perp AB$ if and only if $PS\perp TS$. |
| 2 | Let $n, k$ be given positive integers satisfying $k\le 2n-1$. On a table tennis tournament $2n$ players take part, they play a total of $k$ rounds match, each round is divided into $n$ groups, each group two players match. The two players in different rounds can match on many occasions. Find the greatest positive integer $m=f(n,k)$ such that no matter how the tournament processes, we always find $m$ players each of pair of which didn't match each other. |

**3** Let $x_1, x_2, \cdots, x_m, y_1, y_2, \cdots, y_n$ be positive real numbers. Denote by $X = \sum_{i=1}^{m} x, Y = \sum_{j=1}^{n} y$. Prove that $2XY \sum_{i=1}^{m} \sum_{j=1}^{n} |x_i - y_j| \geq X^2 \sum_{j=1}^{n} \sum_{l=1}^{n} |y_i - y_l| + Y^2 \sum_{i=1}^{m} \sum_{k=1}^{m} |x_i - x_k|$

**–** Quiz 2

**1** In convex pentagon $ABCDE$, denote by $AD \cap BE = F, BE \cap CA = G, CA \cap DB = H, DB \cap EC = I, EC \cap AD = J; AI \cap BE = A', BJ = B', CF = C', DG \cap EC = D', EH \cap AD = E'$. Prove that $\frac{AB'}{B'C} \cdot \frac{CD'}{D'E} \cdot \frac{EA'}{A'B} \cdot \frac{BC'}{C'D} \cdot \frac{DE'}{E'A} = 1$.

**2** Find all the pairs of integers $(a, b)$ satisfying $ab(a-b) \neq 0$ such that there exists a subset $Z_0$ of set of integers $Z$, for any integer $n$, exactly one among three integers $n, n+a, n+b$ belongs to $Z_0$.

**3** Consider function $f : R \to R$ which satisfies the conditions for any mutually distinct real numbers $a, b, c, d$ satisfying $\frac{a-b}{b-c} + \frac{a-d}{d-c} = 0$, $f(a), f(b), f(c), f(d)$ are mutully different and $\frac{f(a)-f(b)}{f(b)-f(c)} + \frac{f(a)-f(d)}{f(d)-f(c)} = 0$. Prove that function $f$ is linear

**–** Quiz 3

**1** Let $\alpha, \beta$ be real numbers satisfying $1 < \alpha < \beta$. Find the greatest positive integer $r$ having the following property: each of positive integers is colored by one of $r$ colors arbitrarily, there always exist two integers $x, y$ having the same color such that $\alpha \leq \frac{x}{y} \leq \beta$.

**2** In convex quadrilateral $ABCD, CB, DA$ are external angle bisectors of $\angle DCA, \angle CDB$, respectively. Points $E, F$ lie on the rays $AC, BD$ respectively such that $CEFD$ is cyclic quadrilateral. Point $P$ lie in the plane of quadrilateral $ABCD$ such that $DA, CB$ are external angle bisectors of $\angle PDE, \angle PCF$ respectively. $AD$ intersects $BC$ at $Q$. Prove that $P$ lies on $AB$ if and only if $Q$ lies on segment $EF$.

**3** Let $f(x)$ be a $n-$degree polynomial all of whose coefficients are equal to $\pm 1$, and having $x = 1$ as its $m$ multiple root. If $m \geq 2^k (k \geq 2, k \in N)$, then $n \geq 2^{k+1} - 1$.

**–** Quiz 4

**1** Given that points $D, E$ lie on the sidelines $AB, BC$ of triangle $ABC$, respectively, point $P$ is in interior of triangle $ABC$ such that $PE = PC$ and $\triangle DEP \sim \triangle PCA$. Prove that $BP$ is tangent of the circumcircle of triangle $PAD$.

**2** Find all integers $n \geq 2$ having the following property: for any $k$ integers $a_1, a_2, \cdots, a_k$ which aren't congruent to each other (modulo $n$), there exists an integer polynomial $f(x)$ such that congruence equation $f(x) \equiv 0 (mod n)$ exactly has $k$ roots $x \equiv a_1, a_2, \cdots, a_k (mod n)$.

 

**3** Let $X$ be a set containing $2k$ elements, $F$ is a set of subsets of $X$ consisting of certain $k$ elements such that any one subset of $X$ consisting of $k-1$ elements is exactly contained in an element of $F$. Show that $k+1$ is a prime number.

**–** Quiz 5

**1** Let $n$ be a composite. Prove that there exists positive integer $m$ satisfying $m|n, m \leq \sqrt{n}$, and $d(n) \leq d^3(m)$. Where $d(k)$ denotes the number of positive divisors of positive integer $k$.

**2** In acute triangle $ABC$, points $P, Q$ lie on its sidelines $AB, AC$, respectively. The circumcircle of triangle $ABC$ intersects of triangle $APQ$ at $X$ (different from $A$). Let $Y$ be the reflection of $X$ in line $PQ$. Given $PX > PB$. Prove that $S_{\triangle XPQ} > S_{\triangle YBC}$. Where $S_{\triangle XYZ}$ denotes the area of triangle $XYZ$.

**3** Let nonnegative real numbers $a_1, a_2, a_3, a_4$ satisfy $a_1+a_2+a_3+a_4 = 1$. Prove that $max\{\sum_1^4 \sqrt{a_i^2 + a_i a_{i-1} + a$
$2.$
Where for all integers $i, a_{i+4} = a_i$ holds.

**–** Quiz 6

**1** Let $a > b > 1, b$ is an odd number, let $n$ be a positive integer. If $b^n | a^n - 1,$ then $a^b > \frac{3^n}{n}$.

**2** Find all complex polynomial $P(x)$ such that for any three integers $a, b, c$ satisfying $a + b + c \neq 0, \frac{P(a)+P(b)+P(c)}{a+b+c}$ is an integer.

**3** Let $(a_n)_{n\geq 1}$ be a sequence of positive integers satisfying $(a_m, a_n) = a_{(m,n)}$ (for all $m, n \in N^+$). Prove that for any $n \in N^+, \prod_{d|n} a_d^{\mu(\frac{n}{d})}$ is an integer. where $d|n$ denotes $d$ take all positive divisors of $n$. Function $\mu(n)$ is defined as follows: if $n$ can be divided by square of certain prime number, then $\mu(1) = 1; \mu(n) = 0$; if $n$ can be expressed as product of $k$ different prime numbers, then $\mu(n) = (-1)^k$.