China Team Selection Test 2010

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– Quiz 1

Day 1

1 Assume real numbers $a_i, b_i\,(i = 0, 1, \cdots, 2n)$ satisfy the following conditions:
(1) for $i = 0, 1, \cdots, 2n-1$, we have $a_i + a_{i+1} \geq 0$;
(2) for $j = 0, 1, \cdots, n-1$, we have $a_{2j+1} \leq 0$;
(2) for any integer p, q, $0 \leq p \leq q \leq n$, we have $\sum_{k=2p}^{2q} b_k > 0$.
Prove that $\sum_{i=0}^{2n} (-1)^i a_i b_i \geq 0$, and determine when the equality holds.

2 Let $ABCD$ be a convex quadrilateral. Assume line AB and CD intersect at E, and B lies between A and E. Assume line AD and BC intersect at F, and D lies between A and F. Assume the circumcircles of $\triangle BEC$ and $\triangle CFD$ intersect at C and P. Prove that $\angle BAP = \angle CAD$ if and only if $BD \parallel EF$.

3 Fine all positive integers $m, n \geq 2$, such that
(1) $m + 1$ is a prime number of type $4k - 1$;
(2) there is a (positive) prime number p and nonnegative integer a, such that

$$\frac{m^{2^n-1} - 1}{m - 1} = m^n + p^a.$$

Day 2

1 Let $\triangle ABC$ be an acute triangle with $AB > AC$, let I be the center of the incircle. Let M, N be the midpoint of AC and AB respectively. D, E are on AC and AB respectively such that $BD \parallel IM$ and $CE \parallel IN$. A line through I parallel to DE intersects BC in P. Let Q be the projection of P on line AI. Prove that Q is on the circumcircle of $\triangle ABC$.

2 Let $M = \{1, 2, \cdots, n\}$, each element of M is colored in either red, blue or yellow. Set $A = \{(x, y, z) \in M \times M \times M | x + y + z \equiv 0 \mod n$, x, y, z are of same color$\}$, $B = \{(x, y, z) \in M \times M \times M | x + y + z \equiv 0 \mod n$, x, y, z are of pairwise distinct color$\}$.
Prove that $2|A| \geq |B|$.

3 Let A be a finite set, and $A_1, A_2, \cdots, A_n$ are subsets of A with the following conditions:
(1) $|A_1| = |A_2| = \cdots = |A_n| = k$, and $k > \frac{|A|}{2}$;

(2) for any $a, b \in A$, there exist $A_r, A_s, A_t\,(1 \le r < s < t \le n)$ such that $a, b \in A_r \cap A_s \cap A_t$;
(3) for any integer $i, j\,(1 \le i < j \le n)$, $|A_i \cap A_j| \le 3$.
Find all possible value(s) of n when k attains maximum among all possible systems $(A_1, A_2, \cdots, A_n, A)$.

– Quiz 2

Day 1

1 Let $ABCD$ be a convex quadrilateral with A, B, C, D concyclic. Assume $\angle ADC$ is acute and $\frac{AB}{BC} = \frac{DA}{CD}$. Let Γ be a circle through A and D, tangent to AB, and let E be a point on Γ and inside $ABCD$.
Prove that $AE \perp EC$ if and only if $\frac{AE}{AB} - \frac{ED}{AD} = 1$.

2 Given positive integer n, find the largest real number $\lambda = \lambda(n)$, such that for any degree n polynomial with complex coefficients $f(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_0$,
and any permutation $x_0, x_1, \cdots, x_n$ of $0, 1, \cdots, n$, the following inequality holds $\sum_{k=0}^{n} |f(x_k) - f(x_{k+1})| \ge \lambda |a_n|$, where $x_{n+1} = x_0$.

3 Let $k > 1$ be an integer, set $n = 2^{k+1}$. Prove that for any positive integers $a_1 < a_2 < \cdots < a_n$, the number $\prod_{1 \le i < j \le n}(a_i + a_j)$ has at least $k+1$ different prime divisors.

Day 2

1 Let $\triangle ABC$ be an acute triangle, and let D be the projection of A on BC. Let M, N be the midpoints of AB and AC respectively. Let Γ_1 and Γ_2 be the circumcircles of $\triangle BDM$ and $\triangle CDN$ respectively, and let K be the other intersection point of Γ_1 and Γ_2. Let P be an arbitrary point on BC and E, F are on AC and AB respectively such that $PEAF$ is a parallelogram. Prove that if MN is a common tangent line of Γ_1 and Γ_2, then K, E, A, F are concyclic.

2 Find all positive real numbers λ such that for all integers $n \ge 2$ and all positive real numbers $a_1, a_2, \cdots, a_n$ with $a_1 + a_2 + \cdots + a_n = n$, the following inequality holds: $\sum_{i=1}^{n} \frac{1}{a_i} - \lambda \prod_{i=1}^{n} \frac{1}{a_i} \le n - \lambda$.

3 For integers $n > 1$, define $f(n)$ to be the sum of all postive divisors of n that are less than n. Prove that for any positive integer k, there exists a positive integer $n > 1$ such that $n < f(n) < f^2(n) < \cdots < f^k(n)$, where $f^i(n) = f(f^{i-1}(n))$ for $i > 1$ and $f^1(n) = f(n)$.

– Quiz 3

Day 1

1 Given integer $n \geq 2$ and positive real number a, find the smallest real number $M = M(n, a)$, such that for any positive real numbers $x_1, x_2, \cdots, x_n$ with $x_1 x_2 \cdots x_n = 1$, the following inequality holds:

$$\sum_{i=1}^{n} \frac{1}{a + S - x_i} \leq M$$

where $S = \sum_{i=1}^{n} x_i$.

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2 In a football league, there are $n \geq 6$ teams. Each team has a homecourt jersey and a road jersey with different color. When two teams play, the home team always wear homecourt jersey and the road team wear their homecourt jersey if the color is different from the home team's homecourt jersey, or otherwise the road team shall wear their road jersey. It is required that in any two games with 4 different teams, the 4 teams' jerseys have at least 3 different color. Find the least number of color that the n teams' $2n$ jerseys may use.

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3 Given positive integer k, prove that there exists a positive integer N depending only on k such that for any integer $n \geq N$, $\binom{n}{k}$ has at least k different prime divisors.

Day 2

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1 Let ω be a semicircle and AB its diameter. ω_1 and ω_2 are two different circles, both tangent to ω and to AB, and ω_1 is also tangent to ω_2. Let P, Q be the tangent points of ω_1 and ω_2 to AB respectively, and P is between A and Q. Let C be the tangent point of ω_1 and ω. Find $\tan \angle ACQ$.

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2 Prove that there exists a sequence of unbounded positive integers $a_1 \leq a_2 \leq a_3 \leq \cdots$, such that there exists a positive integer M with the following property: for any integer $n \geq M$, if $n+1$ is not prime, then any prime divisor of $n! + 1$ is greater than $n + a_n$.

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3 An (unordered) partition P of a positive integer n is an n-tuple of nonnegative integers $P = (x_1, x_2, \cdots, x_n)$ such that $\sum_{k=1}^{n} k x_k = n$. For positive integer $m \leq n$, and a partition $Q = (y_1, y_2, \cdots, y_m)$ of m, Q is called compatible to P if $y_i \leq x_i$ for $i = 1, 2, \cdots, m$. Let $S(n)$ be the number of partitions P of n such that for each odd $m < n$, m has exactly one partition compatible to P and for each even $m < n$, m has exactly two partitions compatible to P. Find $S(2010)$.

– TST 1

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1 Given acute triangle ABC with $AB > AC$, let M be the midpoint of BC. P is a point in triangle AMC such that $\angle MAB = \angle PAC$. Let O, O_1, O_2 be the circumcenters of $\triangle ABC, \triangle ABP, \triangle ACP$ respectively. Prove that line AO passes through the midpoint of $O_1 O_2$.

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2 Let $A = \{a_1, a_2, \cdots, a_{2010}\}$ and $B = \{b_1, b_2, \cdots, b_{2010}\}$ be two sets of complex numbers. Suppose

$$\sum_{1 \leq i < j \leq 2010} (a_i + a_j)^k = \sum_{1 \leq i < j \leq 2010} (b_i + b_j)^k$$

holds for every $k = 1, 2, \cdots, 2010$. Prove that $A = B$.

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3 Let $n_1, n_2, \cdots, n_{26}$ be pairwise distinct positive integers satisfying
(1) for each n_i, its digits belong to the set $\{1, 2\}$;
(2) for each i, j, n_i can't be obtained from n_j by adding some digits on the right.
Find the smallest possible value of $\sum_{i=1}^{26} S(n_i)$, where $S(m)$ denotes the sum of all digits of a positive integer m.

– TST 2

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1 Let $G = G(V, E)$ be a simple graph with vertex set V and edge set E. Suppose $|V| = n$. A map $f : V \to \mathbb{Z}$ is called good, if f satisfies the followings:
(1) $\sum_{v \in V} f(v) = |E|$;
(2) color arbitarily some vertices into red, one can always find a red vertex v such that $f(v)$ is no more than the number of uncolored vertices adjacent to v.
Let $m(G)$ be the number of good maps. Prove that if every vertex in G is adjacent to at least one another vertex, then $n \leq m(G) \leq n!$.

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2 Given integer $a_1 \geq 2$. For integer $n \geq 2$, define a_n to be the smallest positive integer which is not coprime to a_{n-1} and not equal to $a_1, a_2, \cdots, a_{n-1}$. Prove that every positive integer except 1 appears in this sequence $\{a_n\}$.

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3 Given integer $n \geq 2$ and real numbers $x_1, x_2, \cdots, x_n$ in the interval $[0, 1]$. Prove that there exist real numbers $a_0, a_1, \cdots, a_n$ satisfying the following conditions:
(1) $a_0 + a_n = 0$;
(2) $|a_i| \leq 1$, for $i = 0, 1, \cdots, n$;
(3) $|a_i - a_{i-1}| = x_i$, for $i = 1, 2, \cdots, n$.
