China Team Selection Test 2011

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–	Quiz 1
Day 1	
1	In $\triangle ABC$ we have $BC > CA > AB$. The nine point circle is tangent to the incircle, A-excircle, B-excircle and C-excircle at the points T, T_A, T_B, T_C respectively. Prove that the segments TT_B and lines T_AT_C intersect each other.
2	Let S be a set of n points in the plane such that no four points are collinear. Let $\{d_1, d_2, \cdots, d_k\}$ be the set of distances between pairs of distinct points in S, and let m_i be the multiplicity of d_i, i.e. the number of unordered pairs $\{P, Q\} \subseteq S$ with $\lvert PQ\rvert = d_i$. Prove that $\sum_{i=1}^{k} m_i^2 \leq n^3 - n^2$.
3	A positive integer n is known as an *interesting* number if n satisfies $$\{\frac{n}{10^k}\} > \frac{n}{10^{10}}$$ for all $k = 1, 2, \ldots 9$. Find the number of interesting numbers.
Day 2	
1	Let one of the intersection points of two circles with centres O_1, O_2 be P. A common tangent touches the circles at A, B respectively. Let the perpendicular from A to the line BP meet O_1O_2 at C. Prove that $AP \perp PC$.
2	Let n be a positive integer and let α_n be the number of 1's within binary representation of n. Show that for all positive integers r, $$2^{2n-\alpha_n} \quad \Bigg\vert \quad \sum_{k=-n}^{n} \binom{2n}{n+k} k^{2r}.$$
3	For a given integer $n \geq 2$, let $a_0, a_1, \ldots, a_n$ be integers satisfying $0 = a_0 < a_1 < \ldots < a_n = 2n-1$. Find the smallest possible number of elements in the set $\{a_i + a_j \mid 0 \leq i \leq j \leq n\}$.
–	Quiz 2

Day 1

1 Let $n \geq 2$ be a given integer. Find all functions $f:\mathbb{R}\rightarrow \mathbb{R}$ such that

$$f(x-f(y))=f(x+y^n)+f(f(y)+y^n), \qquad \forall x,y \in \mathbb R.$$

2 Let ℓ be a positive integer, and let m,n be positive integers with $m\geq n$, such that $A_1,A_2,\cdots,A_m,B_1,\cdots,B_m$ are $m+n$ pairwise distinct subsets of the set $\{1,2,\cdots,\ell\}$. It is known that $A_i\Delta B_j$ are pairwise distinct, $1\leq i\leq m, 1\leq j\leq n$, and runs over all nonempty subsets of $\{1,2,\cdots,\ell\}$. Find all possible values of m,n.

3 For any positive integer d, prove there are infinitely many positive integers n such that $d(n!)-1$ is a composite number.

Day 2

1 Let AA',BB',CC' be three diameters of the circumcircle of an acute triangle ABC. Let P be an arbitrary point in the interior of $\triangle ABC$, and let D,E,F be the orthogonal projection of P on BC,CA,AB, respectively. Let X be the point such that D is the midpoint of $A'X$, let Y be the point such that E is the midpoint of $B'Y$, and similarly let Z be the point such that F is the midpoint of $C'Z$. Prove that triangle XYZ is similar to triangle ABC.

2 Let $\{b_n\}_{n\geq 1}^{\infty}$ be a sequence of positive integers. The sequence $\{a_n\}_{n\geq 1}^{\infty}$ is defined as follows: a_1 is a fixed positive integer and

$$a_{n+1}=a_n^{b_n}+1 ,\qquad \forall n\geq 1.$$

Find all positive integers $m\geq 3$ with the following property: If the sequence $\{a_n \mod m\}_{n\geq 1}^{\infty}$ is eventually periodic, then there exist positive integers q,u,v with $2\leq q\leq m-1$, such that the sequence $\{b_{v+ut} \mod q\}_{t\geq 1}^{\infty}$ is purely periodic.

3 Let n be a positive integer. Find the largest real number λ such that for all positive real numbers $x_1,x_2,\cdots,x_{2n}$ satisfying the inequality

$$\frac{1}{2n}\sum_{i=1}^{2n}(x_i+2)^n\geq \prod_{i=1}^{2n} x_i,$$

the following inequality also holds

$$\frac{1}{2n}\sum_{i=1}^{2n}(x_i+1)^n\geq \lambda\prod_{i=1}^{2n} x_i.$$

– Quiz 3

Day 1

1 Let $n \geq 3$ be an integer. Find the largest real number M such that for any positive real numbers $x_1, x_2, \cdots, x_n$, there exists an arrangement $y_1, y_2, \cdots, y_n$ of real numbers satisfying

$$\sum_{i=1}^{n} \frac{y_i^2}{y_{i+1}^2 - y_{i+1}y_{i+2} + y_{i+2}^2} \geq M,$$

where $y_{n+1} = y_1, y_{n+2} = y_2$.

2 Let $n > 1$ be an integer, and let k be the number of distinct prime divisors of n. Prove that there exists an integer a, $1 < a < \frac{n}{k} + 1$, such that $n \mid a^2 - a$.

3 Let G be a simple graph with $3n^2$ vertices ($n \geq 2$). It is known that the degree of each vertex of G is not greater than $4n$, there exists at least a vertex of degree one, and between any two vertices, there is a path of length ≤ 3. Prove that the minimum number of edges that G might have is equal to $\frac{(7n^2-3n)}{2}$.

Day 2

1 Let H be the orthocenter of an acute trangle ABC with circumcircle Γ. Let P be a point on the arc BC (not containing A) of Γ, and let M be a point on the arc CA (not containing B) of Γ such that H lies on the segment PM. Let K be another point on Γ such that KM is parallel to the Simson line of P with respect to triangle ABC. Let Q be another point on Γ such that $PQ \parallel BC$. Segments BC and KQ intersect at a point J. Prove that $\triangle KJM$ is an isosceles triangle.

2 Let $a_1, a_2, \ldots, a_n, \ldots$ be any permutation of all positive integers. Prove that there exist infinitely many positive integers i such that $\gcd(a_i, a_{i+1}) \leq \frac{3}{4}i$.

3 Let m and n be positive integers. A sequence of points $(A_0, A_1, \ldots, A_n)$ on the Cartesian plane is called *interesting* if A_i are all lattice points, the slopes of $OA_0, OA_1, \cdots, OA_n$ are strictly increasing (O is the origin) and the area of triangle OA_iA_{i+1} is equal to $\frac{1}{2}$ for $i = 0, 1, \ldots, n-1$.
Let $(B_0, B_1, \cdots, B_n)$ be a sequence of points. We may insert a point B between B_i and B_{i+1} if $\overrightarrow{OB} = \overrightarrow{OB_i} + \overrightarrow{OB_{i+1}}$, and the resulting sequence $(B_0, B_1, \ldots, B_i, B, B_{i+1}, \ldots, B_n)$ is called an *extension* of the original sequence. Given two *interesting* sequences $(C_0, C_1, \ldots, C_n)$ and $(D_0, D_1, \ldots, D_m)$, prove that if $C_0 = D_0$ and $C_n = D_m$, then we may perform finitely many *extensions* on each sequence until the resulting two sequences become identical.