China Team Selection Test 2012
www.artofproblemsolving.com/community/c4968
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– TST 1

Day 1

1 Complex numbers x_i, y_i satisfy $|x_i| = |y_i| = 1$ for $i = 1, 2, \ldots, n$. Let $x = \frac{1}{n}\sum\limits_{i=1}^{n} x_i$, $y = \frac{1}{n}\sum\limits_{i=1}^{n} y_i$ and $z_i = xy_i + yx_i - x_iy_i$. Prove that $\sum\limits_{i=1}^{n} |z_i| \leqslant n$.

2 Given a scalene triangle ABC. Its incircle touches BC, AC, AB at D, E, F respectvely. Let L, M, N be the symmetric points of D with EF,of E with FD,of F with DE,respectively. Line AL intersects BC at P,line BM intersects CA at Q,line CN intersects AB at R. Prove that P, Q, R are collinear.

3 Let $x_n = \binom{2n}{n}$ for all $n \in \mathbb{Z}^+$. Prove there exist infinitely many finite sets A, B of positive integers, satisfying $A \cap B = \emptyset$, and

$$\frac{\prod\limits_{i\in A} x_i}{\prod\limits_{j\in B} x_j} = 2012.$$

Day 2

1 Given two circles ω_1, ω_2, S denotes all ΔABC satisfies that ω_1 is the circumcircle of ΔABC, ω_2 is the A- excircle of ΔABC , ω_2 touches BC, CA, AB at D, E, F. S is not empty, prove that the centroid of ΔDEF is a fixed point.

2 For a positive integer n, denote by $\tau(n)$ the number of its positive divisors. For a positive integer n, if $\tau(m) < \tau(n)$ for all $m < n$, we call n a good number. Prove that for any positive integer k, there are only finitely many good numbers not divisible by k.

3 n being a given integer, find all functions $f\colon \mathbb{Z} \to \mathbb{Z}$, such that for all integers x, y we have $f\left(x + y + f(y)\right) = f(x) + ny$.

– TST 2

Day 1

1 In a simple graph G, we call t pairwise adjacent vertices a t-*clique*. If a vertex is connected with all other vertices in the graph, we call it a *central* vertex. Given are two integers n, k such that $\frac{3}{2} \leq \frac{1}{2}n < k < n$. Let G be a graph on n vertices such that
(1) G does not contain a $(k+1)$-*clique*;
(2) if we add an arbitrary edge to G, that creates a $(k+1)$-*clique*.
Find the least possible number of *central* vertices in G.

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2 Prove that there exists a positive real number C with the following property: for any integer $n \geq 2$ and any subset X of the set $\{1, 2, \ldots, n\}$ such that $|X| \geq 2$, there exist $x, y, z, w \in X$(not necessarily distinct) such that

$$0 < |xy - zw| < C\alpha^{-4}$$

where $\alpha = \frac{|X|}{n}$.

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3 Let $a_1 < a_2$ be two given integers. For any integer $n \geq 3$, let a_n be the smallest integer which is larger than a_{n-1} and can be uniquely represented as $a_i + a_j$, where $1 \leq i < j \leq n-1$. Given that there are only a finite number of even numbers in $\{a_n\}$, prove that the sequence $\{a_{n+1} - a_n\}$ is eventually periodic, i.e. that there exist positive integers T, N such that for all integers $n > N$, we have

$$a_{T+n+1} - a_{T+n} = a_{n+1} - a_n.$$

Day 2

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1 Given an integer $n \geq 2$. Prove that there only exist a finite number of n-tuples of positive integers $(a_1, a_2, \ldots, a_n)$ which simultaneously satisfy the following three conditions:

- $a_1 > a_2 > \ldots > a_n$;
- $\gcd(a_1, a_2, \ldots, a_n) = 1$;
- $a_1 = \sum_{i=1}^{n} \gcd(a_i, a_{i+1})$,where $a_{n+1} = a_1$.

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2 Given two integers m, n which are greater than 1. r, s are two given positive real numbers such that $r < s$. For all $a_{ij} \geq 0$ which are not all zeroes,find the maximal value of the expression

$$f = \frac{(\sum_{j=1}^{n}(\sum_{i=1}^{m} a_{ij}^{s})^{\frac{r}{s}})^{\frac{1}{r}}}{(\sum_{i=1}^{m}) \sum_{j=1}^{n} a_{ij}^{r})^{\frac{s}{r}})^{\frac{1}{s}}}.$$

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3 Given an integer $n \geq 2$, a function $f : \mathbb{Z} \to \{1, 2, \ldots, n\}$ is called *good*, if for any integer $k, 1 \leq k \leq n-1$ there exists an integer $j(k)$ such that for every integer m we have

$$f(m + j(k)) \equiv f(m+k) - f(m) \pmod{n+1}.$$

Find the number of *good* functions.

– TST 3

Day 1

1 In an acute-angled ABC, $\angle A > 60^\circ$, H is its orthocenter. M, N are two points on AB, AC respectively, such that $\angle HMB = \angle HNC = 60^\circ$. Let O be the circumcenter of triangle HMN. D is a point on the same side with A of BC such that $\triangle DBC$ is an equilateral triangle. Prove that H, O, D are collinear.

2 Given an integer $k \geq 2$. Prove that there exist k pairwise distinct positive integers $a_1, a_2, \ldots, a_k$ such that for any non-negative integers $b_1, b_2, \ldots, b_k, c_1, c_2, \ldots, c_k$ satisfying $a_1 \leq b_i \leq 2a_i, i = 1, 2, \ldots, k$ and $\prod_{i=1}^{k} b_i^{c_i} < \prod_{i=1}^{k} b_i$, we have

$$k \prod_{i=1}^{k} b_i^{c_i} < \prod_{i=1}^{k} b_i.$$

3 Find the smallest possible value of a real number c such that for any 2012-degree monic polynomial

$$P(x) = x^{2012} + a_{2011}x^{2011} + \ldots + a_1x + a_0$$

with real coefficients, we can obtain a new polynomial $Q(x)$ by multiplying some of its coefficients by -1 such that every root z of $Q(x)$ satisfies the inequality

$$\left|\operatorname{Im} z\right| \leq c \left|\operatorname{Re} z\right|.$$

Day 2

1 Given an integer $n \geq 4$. $S = \{1, 2, \ldots, n\}$. A, B are two subsets of S such that for every pair of $(a, b), a \in A, b \in B, ab + 1$ is a perfect square. Prove that

$$\min\{|A|, |B|\} \leq \log_2 n.$$

2 Find all integers $k \geq 3$ with the following property: There exist integers m, n such that $1 < m < k$, $1 < n < k$, $\gcd(m, k) = \gcd(n, k) = 1$, $m + n > k$ and $k \mid (m-1)(n-1)$.

3 In some squares of a 2012×2012 grid there are some beetles, such that no square contain more than one beetle. At one moment, all the beetles fly off the grid and then land on the grid again, also satisfying the condition that there is at most one beetle standing in each square.

The vector from the centre of the square from which a beetle B flies to the centre of the square on which it lands is called the *translation vector* of beetle B.
For all possible starting and ending configurations, find the maximum length of the sum of the *translation vectors* of all beetles.