China Team Selection Test 2013

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by Uagu, s372102, sqing, iarnab_kundu

Day 1 March 13th

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1 The quadrilateral $ABCD$ is inscribed in circle ω. F is the intersection point of AC and BD. BA and CD meet at E. Let the projection of F on AB and CD be G and H, respectively. Let M and N be the midpoints of BC and EF, respectively. If the circumcircle of $\triangle MNG$ only meets segment BF at P, and the circumcircle of $\triangle MNH$ only meets segment CF at Q, prove that PQ is parallel to BC.

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2 For the positive integer n, define $f(n) = \min\limits_{m\in\mathbb{Z}}\left|\sqrt{2}-\frac{m}{n}\right|$. Let $\{n_i\}$ be a strictly increasing sequence of positive integers. C is a constant such that $f(n_i) < \dfrac{C}{n_i^2}$ for all $i \in \{1, 2, \ldots\}$. Show that there exists a real number $q > 1$ such that $n_i \geqslant q^{i-1}$ for all $i \in \{1, 2, \ldots\}$.

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3 There are$n$ balls numbered $1, 2, \cdots, n$, respectively. They are painted with 4 colours, red, yellow, blue, and green, according to the following rules:
First, randomly line them on a circle.
Then let any three clockwise consecutive balls numbered i, j, k, in order.
1) If $i > j > k$, then the ball j is painted in red;
2) If $i < j < k$, then the ball j is painted in yellow;
3) If $i < j, k < j$, then the ball j is painted in blue;
4) If $i > j, k > j$, then the ball j is painted in green.
And now each permutation of the balls determine a painting method.
We call two painting methods distinct, if there exists a ball, which is painted with two different colours in that two methods.

Find out the number of all distinct painting methods.

Day 2 March 14th

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1 Let n and k be two integers which are greater than 1. Let $a_1, a_2, \ldots, a_n, c_1, c_2, \ldots, c_m$ be non-negative real numbers such that
i) $a_1 \ge a_2 \ge \ldots \ge a_n$ and $a_1 + a_2 + \ldots + a_n = 1$;
ii) For any integer $m \in \{1, 2, \ldots, n\}$, we have that $c_1 + c_2 + \ldots + c_m \le m^k$.
Find the maximum of $c_1a_1^k + c_2a_2^k + \ldots + c_na_n^k$.

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2 Let P be a given point inside the triangle ABC. Suppose L, M, N are the midpoints of BC, CA, AB

respectively and

$$PL : PM : PN = BC : CA : AB.$$

The extensions of AP, BP, CP meet the circumcircle of ABC at D, E, F respectively. Prove that the circumcentres of $APF, APE, BPF, BPD, CPD, CPE$ are concyclic.

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3 Find all positive real numbers $r < 1$ such that there exists a set $\mathcal{S}$ with the given properties:
i) For any real number t, exactly one of $t, t + r$ and $t + 1$ belongs to $\mathcal{S}$;
ii) For any real number t, exactly one of $t, t - r$ and $t - 1$ belongs to $\mathcal{S}$.

Day 3 March 18th

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1 For a positive integer $k \geq 2$ define $\mathcal{T}_k = \{(x, y) \mid x, y = 0, 1, \dots, k - 1\}$ to be a collection of k^2 lattice points on the cartesian coordinate plane. Let $d_1(k) > d_2(k) > \cdots$ be the decreasing sequence of the distinct distances between any two points in T_k. Suppose $S_i(k)$ be the number of distances equal to $d_i(k)$.
Prove that for any three positive integers $m > n > i$ we have $S_i(m) = S_i(n)$.

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2 Prove that: there exists a positive constant K, and an integer series $\{a_n\}$, satisfying: (1) $0 < a_1 < a_2 < \cdots < a_n < \cdots$; (2) For any positive integer n, $a_n < 1.01^n K$; (3) For any finite number of distinct terms in $\{a_n\}$, their sum is not a perfect square.

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3 Let A be a set consisting of 6 points in the plane. denoted $n(A)$ as the number of the unit circles which meet at least three points of A. Find the maximum of $n(A)$

Day 4 March 19th

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1 For a positive integer $N > 1$ with unique factorization $N = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_k^{\alpha_k}$, we define

$$\Omega(N) = \alpha_1 + \alpha_2 + \cdots + \alpha_k.$$

Let $a_1, a_2, \dots, a_n$ be positive integers and $p(x) = (x + a_1)(x + a_2) \cdots (x + a_n)$ such that for all positive integers k, $\Omega(P(k))$ is even. Show that n is an even number.

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2 Find the greatest positive integer m with the following property:
For every permutation $a_1, a_2, \cdots, a_n, \cdots$ of the set of positive integers, there exists positive integers $i_1 < i_2 < \cdots < i_m$ such that $a_{i_1}, a_{i_2}, \cdots, a_{i_m}$ is an arithmetic progression with an odd common difference.

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3 Let $n > 1$ be an integer and let $a_0, a_1, \dots, a_n$ be non-negative real numbers. Definite $S_k = \sum_{i=0}^{k} \binom{k}{i} a_i$ for $k = 0, 1, \dots, n$. Prove that

$$\frac{1}{n} \sum_{k=0}^{n-1} S_k^2 - \frac{1}{n^2} \left(\sum_{k=0}^{n} S_k \right)^2 \leq \frac{4}{45} (S_n - S_0)^2.$$

Day 5 March 24th

1 Let $n \geq 2$ be an integer. $a_1, a_2, \dots, a_n$ are arbitrarily chosen positive integers with $(a_1, a_2, \dots, a_n) = 1$. Let $A = a_1 + a_2 + \dots + a_n$ and $(A, a_i) = d_i$. Let $(a_2, a_3, \dots, a_n) = D_1, (a_1, a_3, \dots, a_n) = D_2, \dots, (a_1, a_2, \dots, a_{n-1}) = D_n$.
Find the minimum of $\prod_{i=1}^{n} \dfrac{A - a_i}{d_i D_i}$

2 The circumcircle of triangle ABC has centre O. P is the midpoint of $\widehat{BAC}$ and QP is the diameter. Let I be the incentre of $\triangle ABC$ and let D be the intersection of PI and BC. The circumcircle of $\triangle AID$ and the extension of PA meet at F. The point E lies on the line segment PD such that $DE = DQ$. Let R, r be the radius of the inscribed circle and circumcircle of $\triangle ABC$, respectively.
Show that if $\angle AEF = \angle APE$, then $\sin^2 \angle BAC = \dfrac{2r}{R}$

3 101 people, sitting at a round table in any order, had $1, 2, ..., 101$ cards, respectively.
A transfer is someone give one card to one of the two people adjacent to him.
Find the smallest positive integer k such that there always can through no more than k times transfer, each person hold cards of the same number, regardless of the sitting order.

Day 6 March 25th

1 Let p be a prime number and a, k be positive integers such that $p^a < k < 2p^a$. Prove that there exists a positive integer n such that

$$n < p^{2a}, C_n^k \equiv n \equiv k \pmod{p^a}.$$

2 Let $k \geq 2$ be an integer and let $a_1, a_2, \cdots, a_n, b_1, b_2, \cdots, b_n$ be non-negative real numbers. Prove that

$$\left(\frac{n}{n-1}\right)^{n-1} \left(\frac{1}{n} \sum_{i=1}^{n} a_i^2\right) + \left(\frac{1}{n} \sum_{i=1}^{n} b_i\right)^2 \geq \prod_{i=1}^{n} (a_i^2 + b_i^2)^{\frac{1}{n}}.$$

3 A point (x, y) is a *lattice point* if $x, y \in \mathbb{Z}$. Let $E = \{(x, y) : x, y \in \mathbb{Z}\}$. In the coordinate plane, P and Q are both sets of points in and on the boundary of a convex polygon with vertices on lattice points. Let $T = P \cap Q$. Prove that if $T \neq \emptyset$ and $T \cap E = \emptyset$, then T is a non-degenerate convex quadrilateral region.