China Team Selection Test 2014

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– TST 1

Day 1 March 12th

1 $ABCD$ is a cyclic quadrilateral, with diagonals AC, BD perpendicular to each other. Let point F be on side BC, the parallel line EF to AC intersect AB at point E, line FG parallel to BD intersect CD at G. Let the projection of E onto CD be P, projection of F onto DA be Q, projection of G onto AB be R. Prove that QF bisects $\angle PQR$.

2 Let A be a finite set of positive numbers , $B=\{\frac{a+b}{c+d}|a,b,c,d\in A\}$.
Show that: $|B|\ge 2|A|^2-1$,
where $|X|$ be the number of elements of the finite set X.
(High School Affiliated to Nanjing Normal University)

3 Let the function $f:N^*\to N^*$ such that
(1) $(f(m),f(n))\le (m,n)^{2014}, \forall m,n\in N^*$;
(2) $n\le f(n)\le n+2014, \forall n\in N^*$
Show that: there exists the positive integers N such that $f(n)=n$, for each integer $n\ge N$.
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Day 2 March 13th

4 For any real numbers sequence $\{x_n\}$,suppose that $\{y_n\}$ is a sequence such that: $y_1=x_1, y_{n+1}=x_{n+1}-(\sum\limits_{i=1}^{n}x_i^2)^{\frac{1}{2}}$ $(n\ge 1)$.
Find the smallest positive number λ such that for any real numbers sequence $\{x_n\}$ and all positive integers m , have $\frac{1}{m}\sum\limits_{i=1}^{m}x_i^2\le\sum\limits_{i=1}^{m}\lambda^{m-i}y_i^2$.
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5 Let $a_1<a_2<...<a_t$ be t given positive integers where no three form an arithmetic progression. For $k=t,t+1,...$ define a_{k+1} to be the smallest positive integer larger than a_k satisfying the condition that no three of $a_1,a_2,...,a_{k+1}$ form an arithmetic progression. For any $x\in\mathbb{R}^+$ define $A(x)$ to be the number of terms in $\{a_i\}_{i\ge 1}$ that are at most x. Show that there exist $c>1$ and $K>0$ such that $A(x)\ge c\sqrt{x}$ for any $x>K$.

6 Let $n\ge 2$ be a positive integer. Fill up a $n\times n$ table with the numbers $1,2,...,n^2$ exactly once each. Two cells are termed adjacent if they have a common edge. It is known that for any two adjacent cells, the numbers they contain differ by at most n. Show that there exist a 2×2 square of adjacent cells such that the diagonally opposite pairs sum to the same number.

– TST 2

Day 1 March 17th

1 Prove that for any positive integers k and N,

$$\left(\frac{1}{N}\sum_{n=1}^{N}(\omega(n))^k\right)^{\frac{1}{k}}\le k+\sum_{q\le N}\frac{1}{q},$$

where $\sum\limits_{q\le N}\frac{1}{q}$ is the summation over of prime powers $q\le N$ (including $q=1$).
Note: For integer $n>1$, $\omega(n)$ denotes number of distinct prime factors of n, and $\omega(1)=0$.

2 Given a fixed positive integer $a\ge 9$. Prove: There exist finitely many positive integers n, satisfying:
(1)$\tau(n)=a$
(2)$n|\phi(n)+\sigma(n)$
Note: For positive integer n, $\tau(n)$ is the number of positive divisors of n, $\phi(n)$ is the number of positive integers $\le n$ and relatively prime with n, $\sigma(n)$ is the sum of positive divisors of n.

3 A is the set of points of a convex n-gon on a plane. The distinct pairwise distances between any 2 points in A arranged in descending order is $d_1>d_2>...>d_m>0$. Let the number of unordered pairs of points in A such that their distance is d_i be exactly μ_i, for $i=1,2,...,m$. Prove: For any positive integer $k\le m$, $\mu_1+\mu_2+...+\mu_k\le(3k-1)n$.

Day 2 March 18th

4 Given circle O with radius R, the inscribed triangle ABC is an acute scalene triangle, where AB is the largest side. AH_A, BH_B, CH_C are heights on BC, CA, AB. Let D be the symmetric point of H_A with respect to H_BH_C, E be the symmetric point of H_B with respect to H_AH_C. P is the intersection of AD, BE, H is the orthocentre of $\triangle ABC$. Prove: $OP\cdot OH$ is fixed, and find this value in terms of R.

(Edited)

5 Find the smallest positive constant c satisfying: For any simple graph $G=G(V,E)$, if $|E|\ge c|V|$, then G contains 2 cycles with no common vertex, and one of them contains a chord.

Note: The cycle of graph $G(V,E)$ is a set of distinct vertices $v_1,v_2...,v_n\subseteq V$, $v_iv_{i+1}\in E$ for all $1\le i\le n$ $(n\ge 3, v_{n+1}=v_1)$; a cycle containing a chord is the cycle $v_1,v_2...,v_n$, such that there exist $i, j, 1 < i - j < n - 1$, satisfying $v_iv_j \in E$.

6 Let k be a fixed even positive integer, N is the product of k distinct primes $p_1, ..., p_k$, a, b are two positive integers, $a, b \leq N$. Denote $S_1 = \{d | \ d|N, a \leq d \leq b, d$ has even number of prime factors$\}$, $S_2 = \{d | \ d|N, a \leq d \leq b, d$ has odd number of prime factors$\}$,
Prove: $|S_1| - |S_2| \leq C^{\frac{k}{2}}_k$

– TST3

Day 1 March 23rd

1 Let the circumcenter of triangle ABC be O. H_A is the projection of A onto BC. The extension of AO intersects the circumcircle of BOC at A'. The projections of A' onto AB, AC are D, E, and O_A is the circumcentre of triangle DH_AE. Define H_B, O_B, H_C, O_C similarly.
Prove: H_AO_A, H_BO_B, H_CO_C are concurrent

2 Let $A_1A_2...A_{101}$ be a regular 101-gon, and colour every vertex red or blue. Let N be the number of obtuse triangles satisfying the following: The three vertices of the triangle must be vertices of the 101-gon, both the vertices with acute angles have the same colour, and the vertex with obtuse angle have different colour. (1) Find the largest possible value of N. (2) Find the number of ways to colour the vertices such that maximum N is acheived. (Two colourings a different if for some A_i the colours are different on the two colouring schemes).

3 Show that there are no 2-tuples (x, y) of positive integers satisfying the equation $(x+1)(x+2)\cdots(x+2014) = (y+1)(y+2)\cdots(y+4028)$.

Day 2 March 24th

4 Let k be a fixed odd integer, $k > 3$. Prove: There exist infinitely many positive integers n, such that there are two positive integers d_1, d_2 satisfying d_1, d_2 each dividing $\frac{n^2+1}{2}$, and $d_1 + d_2 = n + k$.

5 Let n be a given integer which is greater than 1 . Find the greatest constant $\lambda(n)$ such that for any non-zero complex $z_1, z_2, \cdots, z_n$,have that

$$\sum_{k=1}^{n} |z_k|^2 \geq \lambda(n) \min_{1 \leq k \leq n} \{|z_{k+1} - z_k|^2\},$$

where $z_{n+1} = z_1$.

6 For positive integer $k > 1$, let $f(k)$ be the number of ways of factoring k into product of positive integers greater than 1 (The order of factors are not countered, for example $f(12) = 4$, as 12

can be factored in these 4 ways: $12, 2\cdot 6, 3\cdot 4, 2\cdot 2\cdot 3$.
Prove: If n is a positive integer greater than 1, p is a prime factor of n, then $f(n)\le \frac{n}{p}$