Hong Kong Team Selection Test 2018

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Test 1 August 22, 2017

1 Find all positive integer(s) n such that $n^2 + 32n + 8$ is a perfect square.

2 Find all polynomials f such that f has non-negative integer coefficients, $f(1) = 7$ and $f(2) = 2017.$

3 In a school there are 1200 students. Each student must join exactly k clubs. Given that there is a common club joined by every 23 students, but there is no common club joined by all 1200 students, find the smallest possible value of k.

4 Find infinitely many positive integers m such that for each m, the number $\dfrac{2^{m-1}-1}{8191m}$ is an integer.

5 In a group of 2017 persons, any pair of persons has exactly one common friend (other than the pair of persons). Determine the smallest possible value of the difference between the numbers of friends of the person with the most friends and the person with the least friends in such a group.

6 A triangle ABC has its orthocentre H distinct from its vertices and from the circumcenter O of $\triangle ABC$. Denote by M, N and P respectively the circumcenters of triangles HBC, HCA and HAB. Show that the lines AM, BN, CP and OH are concurrent.

Test 2 October 21, 2017

1 Let ABC be a triangle with $AB = AC$. A circle Γ lies outside triangle ABC and is tangent to line AC at C. Point D lies on Γ such that the circumcircle of triangle ABD is internally tangent to Γ. Segment AD meets Γ secondly at E. Prove that BE is tangent to Γ

2 There are three piles of coins, with a, b and c coins respectively, where $a, b, c \geq 2015$ are positive integers. The following operations are allowed:

(1) Choose a pile with an even number of coins and remove all coins from this pile. Add coins to each of the remaining two piles with amount equal to half of that removed; or

(2) Choose a pile with an odd number of coins and at least 2017 coins. Remove 2017 coins from this pile. Add 1009 coins to each of the remaining two piles.

Suppose there are sufficiently many spare coins. Find all ordered triples (a, b, c) such that after some finite sequence of allowed operations. There exists a pile with at least 2017^{2017} coins.

3 Find all functions $f : \mathbb{R} \to \mathbb{R}$ such that

$$f(f(xy - x)) + f(x + y) = yf(x) + f(y)$$

for all real numbers x and y.

4 In triangle ABC with incentre I, let M_A, M_B and M_C by the midpoints of BC, CA and AB respectively, and H_A, H_B and H_C be the feet of the altitudes from A, B and C to the respective sides. Denote by ℓ_b the line being tangent tot he circumcircle of triangle ABC and passing through B, and denote by ℓ'_b the reflection of ℓ_b in BI. Let P_B by the intersection of M_AM_C and ℓ_b, and let Q_B be the intersection of H_AH_C and ℓ'_b. Defined $\ell_c, \ell'_c, P_C, Q_C$ analogously. If R is the intersection of P_BQ_B and P_CQ_C, prove that $RB = RC$.

Test 3 April 29, 2018

1 Does there exist a polynomial $P(x)$ with integer coefficients such that $P(1 + \sqrt[3]{2}) = 1 + \sqrt[3]{2}$ and $P(1 + \sqrt{5}) = 2 + 3\sqrt{5}$?

2 Given triangle ABC, let D be an inner point of segment BC. Let P and Q be distinct inner points of the segment AD. Let $K = BP \cap AC, L = CP \cap AB, E = BQ \cap AC, F = CQ \cap AB$. Given that $KL \parallel EF$, find all possible values of the ratio $BD : DC$.

3 On a rectangular board with m rows and n columns, where $m \leq n$, some squares are coloured black in a way that no two rows are alike. Find the biggest integer k such that for every possible colouring to start with one can always color k columns entirely red in such a way that no two rows are still alike.

Test 4 May 1, 2018

1 The altitudes AD and BE of acute triangle ABC intersect at H. Let F be the intersection of AB and a line that is parallel to the side BC and goes through the circumcentre of ABC. Let M be the midpoint of AH. Prove that $\angle CMF = 90^\circ$

2 For which natural number n is it possible to place natural number from 1 to $3n$ on the edges of a right n-angled prism (on each edge there is exactly one number placed and each one is used exactly 1 time) in such a way, that the sum of all the numbers, that surround each face is the same?

3 Find all primes p and all positive integers a and m such that $a \leq 5p^2$ and $(p-1)! + a = p^m$