India International Mathematical Olympiad Training Camp 2001

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Day 1

1 Let x , y , $z > 0$. Prove that if $xyz \geq xy + yz + zx$, then $xyz \geq 3(x + y + z)$.

2 Two symbols A and B obey the rule $ABBB = B$. Given a word $x_1x_2 \ldots x_{3n+1}$ consisting of n letters A and $2n + 1$ letters B, show that there is a unique cyclic permutation of this word which reduces to B.

3 In a triangle ABC with incircle ω and incenter I , the segments AI , BI , CI cut ω at D , E , F , respectively. Rays AI , BI , CI meet the sides BC , CA , AB at L , M , N respectively. Prove that:

$$AL + BM + CN \leq 3(AD + BE + CF)$$

When does equality occur?

Day 2

1 For any positive integer n, show that there exists a polynomial $P(x)$ of degree n with integer coefficients such that $P(0), P(1), \ldots, P(n)$ are all distinct powers of 2.

2 Let $Q(x)$ be a cubic polynomial with integer coefficients. Suppose that a prime p divides $Q(x_j)$ for $j = 1$,2 ,3 ,4 , where x_1, x_2, x_3, x_4 are distinct integers from the set $\{0, 1, \cdots, p - 1\}$. Prove that p divides all the coefficients of $Q(x)$.

3 Find the number of all unordered pairs $\{A, B\}$ of subsets of an 8-element set, such that $A \cap B \neq \emptyset$ and $|A| \neq |B|$.

Day 3

1 If on $\triangle ABC$, trinagles AEB and AFC are constructed externally such that $\angle AEB = 2\alpha$, $\angle AFB = 2\beta$. $AE = EB$, $AF = FC$.
COnstructed externally on BC is triangle BDC with $\angle DBC = \beta$, $\angle BCD = \alpha$.
Prove that 1. DA is perpendicular to EF.
2. If T is the projection of D on BC, then prove that $\frac{DA}{EF} = 2\frac{DT}{BC}$.

2 Find all functions $f : \mathbb{R}_+ \to \mathbb{R}_+$ satisfying :

$$f(f(x) - x) = 2x$$

for all $x > 0$.

3 Points $B = B_1, B_2, \cdots, B_n, B_{n+1} = C$ are chosen on side BC of a triangle ABC in that order. Let r_j be the inradius of triangle AB_jB_{j+1} for $j = 1, \cdots, n$, and r be the inradius of $\triangle ABC$. Show that there is a constant λ independent of n such that :

$$(\lambda - r_1)(\lambda - r_2)\cdots(\lambda - r_n) = \lambda^{n-1}(\lambda - r)$$

Day 4

1 Complex numbers α , β , γ have the property that $\alpha^k + \beta^k + \gamma^k$ is an integer for every natural number k. Prove that the polynomial

$$(x - \alpha)(x - \beta)(x - \gamma)$$

has integer coefficients.

2 Let $p > 3$ be a prime. For each $k \in \{1, 2, \ldots, p-1\}$, define x_k to be the unique integer in $\{1, \ldots, p-1\}$ such that $kx_k \equiv 1 \pmod{p}$ and set $kx_k = 1 + pn_k$. Prove that :

$$\sum_{k=1}^{p-1} kn_k \equiv \frac{p-1}{2} \pmod{p}$$

3 Each vertex of an $m \times n$ grid is colored blue, green or red in such a way that all the boundary vertices are red. We say that a unit square of the grid is properly colored if: (i) all the three colors occur at the vertices of the square, and (ii) one side of the square has the endpoints of the same color.
Show that the number of properly colored squares is even.

Day 5

1 Let $ABCD$ be a rectangle, and let ω be a circular arc passing through the points A and C.
Let ω_1 be the circle tangent to the lines CD and DA and to the circle ω, and lying completely inside the rectangle $ABCD$.
Similiarly let ω_2 be the circle tangent to the lines AB and BC and to the circle ω, and lying completely inside the rectangle $ABCD$.
Denote by r_1 and r_2 the radii of the circles ω_1 and ω_2, respectively, and by r the inradius of triangle ABC.
(a) Prove that $r_1 + r_2 = 2r$.
(b) Prove that one of the two common internal tangents of the two circles ω_1 and ω_2 is parallel to the line AC and has the length $|AB - AC|$.

2 A strictly increasing sequence (a_n) has the property that $\gcd(a_m, a_n) = a_{\gcd(m,n)}$ for all $m, n \in \mathbb{N}$. Suppose k is the least positive integer for which there exist positive integers $r < k < s$ such that $a_k^2 = a_r a_s$. Prove that $r|k$ and $k|s$.

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3 Let $P(x)$ be a polynomial of degree n with real coefficients and let $a \geq 3$. Prove that

$$\max_{0 \leq j \leq n+1} \left| a^j - P(j) \right| \geq 1$$
