India International Mathematical Olympiad Training Camp 2003

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1 Let A', B', C' be the midpoints of the sides BC, CA, AB, respectively, of an acute non-isosceles triangle ABC, and let D, E, F be the feet of the altitudes through the vertices A, B, C on these sides respectively. Consider the arc DA' of the nine point circle of triangle ABC lying outside the triangle. Let the point of trisection of this arc closer to A' be A''. Define analogously the points B'' (on arc EB') and C''(on arc FC'). Show that triangle $A''B''C''$ is equilateral.

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2 Find all triples (a, b, c) of positive integers such that
(i) $a \leq b \leq c$;
(ii) gcd$(a, b, c) = 1$; and
(iii) $a^3 + b^3 + c^3$ is divisible by each of the numbers a^2b, b^2c, c^2a.

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3 Find all functions $f : \mathbb{R} \to \mathbb{R}$ such that for all reals x and y,

$$f(x + y) + f(x)f(y) = f(xy) + f(x) + f(y).$$

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4 There are four lines in the plane, no three concurrent, no two parallel, and no three forming an equilateral triangle. If one of them is parallel to the Euler line of the triangle formed by the other three lines, prove that a similar statement holds for each of the other lines.

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5 On the real number line, paint red all points that correspond to integers of the form $81x + 100y$, where x and y are positive integers. Paint the remaining integer point blue. Find a point P on the line such that, for every integer point T, the reflection of T with respect to P is an integer point of a different colour than T.

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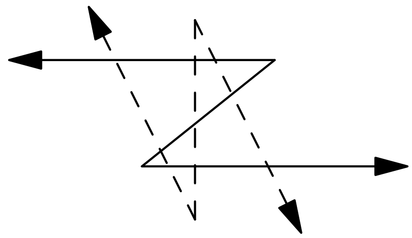
6 A zig-zag in the plane consists of two parallel half-lines connected by a line segment. Find z_n, the maximum number of regions into which n zig-zags can divide the plane. For example, $z_1 = 2, z_2 = 12$(see the diagram). Of these z_n regions how many are bounded? [The zig-zags can be as narrow as you please.] Express your answers as polynomials in n of degree not exceeding 2.



7 p is a polynomial with integer coefficients and for every natural n we have $p(n) > n$. x_k is a sequence that: $x_1 = 1, x_{i+1} = p(x_i)$ for every N one of x_i is divisible by N. Prove that $p(x) = x + 1$

8 Let ABC be a triangle, and let r, r_1, r_2, r_3 denoted its inradius and the exradii opposite the vertices A, B, C, respectively. Suppose $a > r_1, b > r_2, c > r_3$. Prove that
(a) triangle ABC is acute,
(b) $a + b + c > r + r_1 + r_2 + r_3$.

9 Let n be a positive integer and $\{A, B, C\}$ a partition of $\{1, 2, \ldots, 3n\}$ such that $|A| = |B| = |C| = n$. Prove that there exist $x \in A$, $y \in B$, $z \in C$ such that one of x, y, z is the sum of the other two.

10 Let n be a positive integer greater than 1, and let p be a prime such that n divides $p - 1$ and p divides $n^3 - 1$. Prove that $4p - 3$ is a square.