India International Mathematical Olympiad Training Camp 2004

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– Practice Tests

Day 1

1 Let $ABCD$ be a cyclic quadrilateral. Let P, Q, R be the feet of the perpendiculars from D to the lines BC, CA, AB, respectively. Show that $PQ = QR$ if and only if the bisectors of $\angle ABC$ and $\angle ADC$ are concurrent with AC.

2 Prove that for every positive integer n there exists an n-digit number divisible by 5^n all of whose digits are odd.

3 For a, b, c positive reals find the minimum value of

$$\frac{a^2+b^2}{c^2+ab}+\frac{b^2+c^2}{a^2+bc}+\frac{c^2+a^2}{b^2+ca}.$$

4 Given a permutation $\sigma = (a_1, a_2, a_3, ...a_n)$ of $(1, 2, 3, ...n)$, an ordered pair (a_j, a_k) is called an inversion of σ if $a \leq j < k \leq n$ and $a_j > a_k$. Let $m(\sigma)$ denote the no. of inversions of the permutation σ. Find the average of $m(\sigma)$ as σ varies over all permutations.

Day 2

1 Prove that in any triangle ABC,

$$0 < \cot\left(\frac{A}{4}\right) - \tan\left(\frac{B}{4}\right) - \tan\left(\frac{C}{4}\right) - 1 < 2\cot\left(\frac{A}{2}\right).$$

2 Find all triples (x, y, n) of positive integers such that

$$(x+y)(1+xy) = 2^n$$

3 Suppose the polynomial $P(x) \equiv x^3 + ax^2 + bx + c$ has only real zeroes and let $Q(x) \equiv 5x^2 - 16x + 2004$. Assume that $P(Q(x)) = 0$ has no real roots. Prove that $P(2004) > 2004$

4 Let f be a bijection of the set of all natural numbers on to itself. Prove that there exists positive integers $a < a+d < a+2d$ such that $f(a) < f(a+d) < f(a+2d)$

– Selection Tests

Day 1

1 A set A_1, A_2, A_3, A_4 of 4 points in the plane is said to be *Athenian* set if there is a point P of the plane satsifying

(*) P does not lie on any of the lines A_iA_j for $1 \leq i < j \leq 4$;
(**) the line joining P to the mid-point of the line A_iA_j is perpendicular to the line joining P to the mid-point of A_kA_l, i, j, k, l being distinct.

(a) Find all *Athenian* sets in the plane.
(b) For a given *Athenian* set, find the set of all points P in the plane satisfying (*) and (**)

2 Determine all integers a such that $a^k + 1$ is divisible by 12321 for some k

3 The game of $pebbles$ is played on an infinite board of lattice points (i, j). Initially there is a $pebble$ at $(0, 0)$. A move consists of removing a $pebble$ from point $(i, j)$and placing a $pebble$ at each of the points $(i+1, j)$ and $(i, j+1)$ provided both are vacant. Show taht at any stage of the game there is a $pebble$ at some lattice point (a, b) with $0 \leq a+b \leq 3$

Day 2

1 Let ABC be a triangle and let P be a point in its interior. Denote by D, E, F the feet of the perpendiculars from P to the lines BC, CA, AB, respectively. Suppose that

$$AP^2 + PD^2 = BP^2 + PE^2 = CP^2 + PF^2.$$

Denote by I_A, I_B, I_C the excenters of the triangle ABC. Prove that P is the circumcenter of the triangle $I_AI_BI_C$.

Proposed by C.R. Pranesachar, India

2 Show that the only solutions of te equation

$$p^k + 1 = q^m$$

, in positive integers $k, q, m > 1$ and prime p are
(i) $(p, k, q, m) = (2, 3, 3, 2)$
(ii) $k = 1, q = 2$,and p is a prime of the form $2^m - 1$, $m > 1 \in \mathbb{N}$

3 Determine all functionf $f : \mathbb{R} \mapsto \mathbb{R}$ such that

$$f(x+y) = f(x)f(y) - c\sin x \sin y$$

for all reals x, y where $c > 1$ is a given constant.

Day 3

1 Let ABC be a triangle and I its incentre. Let ϱ_1 and ϱ_2 be the inradii of triangles IAB and IAC respectively.
(a) Show that there exists a function $f : (0, \pi) \mapsto \mathbb{R}$ such that

$$\frac{\varrho_1}{\varrho_2} = \frac{f(C)}{f(B)}$$

where $B = \angle ABC$ and $C = \angle BCA$
(b) Prove that

$$2(\sqrt{2}-1) < \frac{\varrho_1}{\varrho_2} < \frac{1+\sqrt{2}}{2}$$

2 Define a function $g : \mathbb{N} \mapsto \mathbb{N}$ by the following rule:
(a) g is nondecrasing
(b) for each n, $g(n)$ i sthe number of times n appears in the range of g,

Prove that $g(1) = 1$ and $g(n+1) = 1 + g(n+1-g(g(n)))$ for all $n \in \mathbb{N}$

3 Two runners start running along a circular track of unit length from the same starting point and int he same sense, with constant speeds v_1 and v_2 respectively, where v_1 and v_2 are two distinct relatively prime natural numbers. They continue running till they simultneously reach the starting point. Prove that

(a) at any given time t, at least one of the runners is at a distance not more than $\frac{[\frac{v_1+v_2}{2}]}{v_1+v_2}$ units from the starting point.
(b) there is a time t such that both the runners are at least $\frac{[\frac{v_1+v_2}{2}]}{v_1+v_2}$ units away from the starting point. (All disstances are measured along the track). $[x]$ is the greatest integer function.

Day 4

1 Let $x_1, x_2, x_3, ...x_n$ be n real numbers such that $0 < x_j < \frac{1}{2}$. Prove that

$$\frac{\prod_{j=1}^{n} x_j}{\left(\sum_{j=1}^{n} x_j\right)^n} \leq \frac{\prod_{j=1}^{n} (1 - x_j)}{\left(\sum_{j=1}^{n} (1 - x_j)\right)^n}$$

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2 Find all primes $p \geq 3$ with the following property: for any prime $q < p$, the number

$$p - \left\lfloor \frac{p}{q} \right\rfloor q$$

is squarefree (i.e. is not divisible by the square of a prime).

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3 Every point with integer coordinates in the plane is the center of a disk with radius $1/1000$.

(1) Prove that there exists an equilateral triangle whose vertices lie in different discs.

(2) Prove that every equilateral triangle with vertices in different discs has side-length greater than 96.

Radu Gologan, Romania

The "¿ 96" in **(b)** can be strengthened to "¿ 124". By the way, part **(a)** of this problem is the place where I used the well-known "Dedekind" theorem (`http://mathlinks.ro/viewtopic.php?t=5537`).

Day 5

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1 Let ABC be an acute-angled triangle and Γ be a circle with AB as diameter intersecting BC and CA at $F(\neq B)$ and $E(\neq A)$ respectively. Tangents are drawn at E and F to Γ intersect at P. Show that the ratio of the circumcentre of triangle ABC to that if EFP is a rational number.

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2 Let $P(x) = x^4 + ax^3 + bx^2 + cx + d$ and $Q(x) = x^2 + px + q$be two real polynomials. Suppose that there exista an interval (r, s) of length greater than 2 SUCH THAT BOTH $P(x)$ AND $Q(x)$ ARE nEGATIVE FOR $X \in (r, s)$ and both are positive for $x > s$ and $x < r$. Show that there is a real x_0 such that $P(x_0) < Q(x_0)$

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3 An integer n is said to be *good* if $|n|$ is not the square of an integer. Determine all integers m with the following property: m can be represented, in infinitely many ways, as a sum of three distinct good integers whose product is the square of an odd integer.

Proposed by Hojoo Lee, Korea
