India International Mathematical Olympiad Training Camp 2005

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by darij grinberg, Rushil, Tales, vinoth_90_2004, orl, Pascual2005, ZetaX, pigfly, pbornsztein

Day 1

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1 Let ABC be a triangle with all angles $\leq 120^\circ$. Let F be the Fermat point of triangle ABC, that is, the interior point of ABC such that $\angle AFB = \angle BFC = \angle CFA = 120^\circ$. For each one of the three triangles BFC, CFA and AFB, draw its Euler line - that is, the line connecting its circumcenter and its centroid.

Prove that these three Euler lines pass through one common point.

Remark. The Fermat point F is also known as the **first Fermat point** or the **first Toricelli point** of triangle ABC.

Floor van Lamoen

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2 Prove that one can find a $n_0 \in \mathbb{N}$ such that $\forall m \geq n_0$, there exist three positive integers a, b , c such that

(i) $m^3 < a < b < c < (m+1)^3$;

(ii) abc is the cube of an integer.

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3 If a, b ,c are three positive real numbers such that $ab + bc + ca = 1$, prove that

$$\sqrt[3]{\frac{1}{a} + 6b} + \sqrt[3]{\frac{1}{b} + 6c} + \sqrt[3]{\frac{1}{c} + 6a} \leq \frac{1}{abc}.$$

Day 2

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1 Consider a n-sided polygon inscribed in a circle ($n \geq 4$). Partition the polygon into $n-2$ triangles using **non-intersecting** diagnols. Prove that, irrespective of the triangulation, the sum of the in-radii of the triangles is a constant.

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2 Let $\tau(n)$ denote the number of positive divisors of the positive integer n. Prove that there exist infinitely many positive integers a such that the equation $\tau(an) = n$ does not have a positive integer solution n.

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3 There are 10001 students at an university. Some students join together to form several clubs (a student may belong to different clubs). Some clubs join together to form several societies (a club may belong to different societies). There are a total of k societies. Suppose that the following conditions hold:

i.) Each pair of students are in exactly one club.

ii.) For each student and each society, the student is in exactly one club of the society.

iii.) Each club has an odd number of students. In addition, a club with $2m+1$ students (m is a positive integer) is
in exactly m societies.

Find all possible values of k.

Proposed by Guihua Gong, Puerto Rico

Day 3

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1 Let $0 < a < b$ be two rational numbers. Let M be a set of positive real numbers with the properties:

(i) $a \in M$ and $b \in M$;

(ii) if $x \in M$ and $y \in M$, then $\sqrt{xy} \in M$.

Let M^*denote the set of all irrational numbers in M. prove that every c, d such that $a < c < d < b$, M^* contains an element m with property $c < m < d$

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2 Find all functions $f : \mathbb{N}^* \to \mathbb{N}^*$ satisfying

$$\left(f^2\left(m\right)+f\left(n\right)\right) \mid \left(m^2+n\right)^2$$

for any two positive integers m and n.

Remark. The abbreviation $\mathbb{N}^*$ stands for the set of all positive integers: $\mathbb{N}^* = \{1, 2, 3, ...\}$.
By $f^2\left(m\right)$, we mean $\left(f\left(m\right)\right)^2$ (and not $f\left(f\left(m\right)\right)$).

Proposed by Mohsen Jamali, Iran

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3 A merida path of order $2n$ is a lattice path in the first quadrant of xy- plane joining $(0,0)$ to $(2n,0)$ using three kinds of steps $U=(1,1)$, $D=(1,-1)$ and $L=(2,0)$, i.e. U joins $x,y)$ to $(x+1,y+1)$ etc... An ascent in a merida path is a maximal string of consecutive steps of the form U. If $S(n,k)$ denotes the number of merdia paths of order $2n$ with exactly k ascents, compute $S(n,1)$ and $S(n,n-1)$.

Day 4

1 Let $ABCD$ be a convex quadrilateral. The lines parallel to AD and CD through the orthocentre H of ABC intersect AB and BC Crespectively at P and Q. prove that the perpendicular through H to th eline PQ passes through th eorthocentre of triangle ACD

2 Given real numbers $a, \alpha, \beta, \sigma\ and\ \varrho$ s.t. $\sigma, \varrho > 0$ and $\sigma\varrho = \frac{1}{16}$, prove that there exist integers x and y s.t.

$$-\sigma \leq (x + \alpha_(ax + y + \beta) \leq \varrho$$

3 Consider a matrix of size $n \times n$ whose entries are real numbers of absolute value not exceeding 1. The sum of all entries of the matrix is 0. Let n be an even positive integer. Determine the least number C such that every such matrix necessarily has a row or a column with the sum of its entries not exceeding C in absolute value.

Proposed by Marcin Kuczma, Poland

Day 5

1 For a given triangle ABC, let X be a variable point on the line BC such that the point C lies between the points B and X. Prove that the radical axis of the incircles of the triangles ABX and ACX passes through a point independent of X.

This is a slight extension of the IMO Shortlist 2004 geometry problem 7 (`http://www.mathlinks.ro/Forum/viewtopic.php?t=41033`) and can be found, together with the proposed solution, among the files uploaded at http://www.mathlinks.ro/Forum/viewtopic.php?t=15622 . Note that the problem was proposed by Russia. I could not find the names of the authors, but I have two particular persons under suspicion. Maybe somebody could shade some light on this...

Darij

2 Determine all positive integers $n > 2$, such that

$$\frac{1}{2}\varphi(n) \equiv 1 (\bmod 6)$$

3 For real numbers a, b, c, d not all equal to 0 , define a real function $f(x) = a + b\cos 2x + c\sin 5x + d\cos 8x$. Suppose $f(t) = 4a$ for some real t. prove that there exist a real number s s.t. $f(s) < 0$