India International Mathematical Olympiad Training Camp 2006

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Day 1

1 Let n be a positive integer divisible by 4. Find the number of permutations σ of $(1,2,3,\cdots,n)$ which satisfy the condition $\sigma(j)+\sigma^{-1}(j)=n+1$ for all $j\in\{1,2,3,\cdots,n\}$.

2 Let $ABCD$ be a parallelogram. A variable line g through the vertex A intersects the rays BC and DC at the points X and Y, respectively. Let K and L be the A-excenters of the triangles ABX and ADY. Show that the angle $\measuredangle KCL$ is independent of the line g.

Proposed by Vyacheslev Yasinskiy, Ukraine

3 There are n markers, each with one side white and the other side black. In the beginning, these n markers are aligned in a row so that their white sides are all up. In each step, if possible, we choose a marker whose white side is up (but not one of the outermost markers), remove it, and reverse the closest marker to the left of it and also reverse the closest marker to the right of it. Prove that, by a finite sequence of such steps, one can achieve a state with only two markers remaining if and only if $n-1$ is not divisible by 3.

Proposed by Dusan Dukic, Serbia

Day 2

1 Let ABC be a triangle and let P be a point in the plane of ABC that is inside the region of the angle BAC but outside triangle ABC.

(a) Prove that any two of the following statements imply the third.

(i) the circumcentre of triangle PBC lies on the ray $\vec{PA}$.

(ii) the circumcentre of triangle CPA lies on the ray $\vec{PB}$.

(iii) the circumcentre of triangle APB lies on the ray $\vec{PC}$.

(b) Prove that if the conditions in (a) hold, then the circumcentres of triangles BPC,CPA and APB lie on the circumcircle of triangle ABC.

2 Let p be a prime number and let X be a finite set containing at least p elements. A collection of pairwise mutually disjoint p-element subsets of X is called a p-family. (In particular, the empty collection is a p-family.) Let A(respectively, B) denote the number of p-families having an even (respectively, odd) number of p-element subsets of X. Prove that A and B differ by a multiple of p.

3 Let ABC be an equilateral triangle, and let D, E and F be points on BC, BA and AB respectively. Let $\angle BAD = \alpha, \angle CBE = \beta$ and $\angle ACF = \gamma$. Prove that if $\alpha + \beta + \gamma \geq 120^\circ$, then the union of the triangular regions BAD, CBE, ACF covers the triangle ABC.

Day 3

1 Let ABC be a triangle with inradius r, circumradius R, and with sides $a = BC, b = CA, c = AB$. Prove that

$$\frac{R}{2r} \geq \left(\frac{64a^2b^2c^2}{(4a^2 - (b-c)^2)(4b^2 - (c-a)^2)(4c^2 - (a-b)^2)} \right)^2.$$

2 the positive divisors $d_1, d_2, \cdots, d_k$ of a positive integer n are ordered

$$1 = d_1 < d_2 < \cdots < d_k = n$$

Suppose $d_7^2 + d_{15}^2 = d_{16}^2$. Find all possible values of d_{17}.

3 Let $A_1, A_2, \cdots, A_n$ be arithmetic progressions of integers, each of k terms, such that any two of these arithmetic progressions have at least two common elements. Suppose b of these arithmetic progressions have common difference d_1 and the remaining arithmetic progressions have common difference d_2 where $0 < b < n$. Prove that

$$b \leq 2 \left(k - \frac{d_2}{gcd(d_1, d_2)} \right) - 1.$$

Day 4

1 Find all triples (a, b, c) such that a, b, c are integers in the set $\{2000, 2001, \ldots, 3000\}$ satisfying $a^2 + b^2 = c^2$ and gcd$(a, b, c) = 1$.

2 Let u_{jk} be a real number for each $j = 1, 2, 3$ and each $k = 1, 2$ and let N be an integer such that

$$\max_{1 \leq k \leq 2} \sum_{j=1}^{3} |u_{jk}| \leq N$$

Let M and l be positive integers such that $l^2 < (M+1)^3$. Prove that there exist integers ξ_1, ξ_2, ξ_3 not all zero, such that

$$\max_{1 \leq j \leq 3} \xi_j \leq M \quad \text{and} \quad \left| \sum_{j=1}^{3} u_{jk}\xi_k \right| \leq \frac{MN}{l} \quad \text{for k=1,2}$$

3 Let $A_1, A_2, \ldots, A_n$ be subsets of a finite set S such that $|A_j| = 8$ for each j. For a subset B of S let $F(B) = \{j \mid 1 \le j \le n \text{ and } A_j \subset B\}$. Suppose for each subset B of S at least one of the following conditions holds

(a) $|B| > 25$,

(b) $F(B) = \emptyset$,

(c) $\bigcap_{j \in F(B)} A_j \neq \emptyset$.

Prove that $A_1 \cap A_2 \cap \cdots \cap A_n \neq \emptyset$.

