India International Mathematical Olympiad Training Camp 2007

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Day 1

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1 Show that in a non-equilateral triangle, the following statements are equivalent: (a) The angles of the triangle are in arithmetic progression. (b) The common tangent to the Nine-point circle and the Incircle is parallel to the Euler Line.

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2 Find all integer solutions of the equation

$$\frac{x^7-1}{x-1}=y^5-1.$$

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3 Let $\mathbb{X}$ be the set of all bijective functions from the set $S=\{1,2,\cdots,n\}$ to itself. For each $f\in\mathbb{X}$, define

$$T_f(j)=\begin{cases}1, & \text{if } f^{(12)}(j)=j,\\ 0, & \text{otherwise}\end{cases}$$

Determine $\sum_{f\in\mathbb{X}}\sum_{j=1}^{n}T_f(j)$.
(Here $f^{(k)}(x)=f(f^{(k-1)}(x))$ for all $k\geq 2$.)

Day 2

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1 Let $ABCD$ be a trapezoid with parallel sides $AB>CD$. Points K and L lie on the line segments AB and CD, respectively, so that $AK/KB=DL/LC$. Suppose that there are points P and Q on the line segment KL satisfying

$$\angle APB=\angle BCD \qquad \text{and} \qquad \angle CQD=\angle ABC.$$

Prove that the points P, Q, B and C are concyclic.

Proposed by Vyacheslev Yasinskiy, Ukraine

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2 Let a,b,c be non-negative real numbers such that $a+b\leq c+1, b+c\leq a+1$ and $c+a\leq b+1$. Show that

$$a^2+b^2+c^2\leq 2abc+1.$$

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3 Given a finite string S of symbols X and O, we denote $\Delta(s)$ as the number of X's in S minus the number of O's (For example, $\Delta(XOOXOOX) = -1$). We call a string S **balanced** if every sub-string T of (consecutive symbols) S has the property $-1 \leq \Delta(T) \leq 2$. (Thus $XOOXOOX$ is not balanced, since it contains the sub-string $OOXOO$ whose Δ value is -3. Find, with proof, the number of balanced strings of length n.

Day 3

1 A sequence of real numbers $a_0,\ a_1,\ a_2,\dots$ is defined by the formula

$$a_{i+1} = \lfloor a_i \rfloor \cdot \langle a_i \rangle \qquad \text{for} \quad i \geq 0;$$

here a_0 is an arbitrary real number, $\lfloor a_i \rfloor$ denotes the greatest integer not exceeding a_i, and $\langle a_i \rangle = a_i - \lfloor a_i \rfloor$. Prove that $a_i = a_{i+2}$ for i sufficiently large.

Proposed by Harmel Nestra, Estionia

2 Let S be a finite set of points in the plane such that no three of them are on a line. For each convex polygon P whose vertices are in S, let $a(P)$ be the number of vertices of P, and let $b(P)$ be the number of points of S which are outside P. A line segment, a point, and the empty set are considered as convex polygons of 2, 1, and 0 vertices respectively. Prove that for every real number x

$$\sum_P x^{a(P)}(1-x)^{b(P)} = 1,$$

where the sum is taken over all convex polygons with vertices in S.

Alternative formulation:

Let M be a finite point set in the plane and no three points are collinear. A subset A of M will be called round if its elements is the set of vertices of a convex $A-$gon $V(A)$. For each round subset let $r(A)$ be the number of points from M which are exterior from the convex $A-$gon $V(A)$. Subsets with $0, 1$ and 2 elements are always round, its corresponding polygons are the empty set, a point or a segment, respectively (for which all other points that are not vertices of the polygon are exterior). For each round subset A of M construct the polynomial

$$P_A(x) = x^{|A|}(1-x)^{r(A)}.$$

Show that the sum of polynomials for all round subsets is exactly the polynomial $P(x) = 1$.

Proposed by Federico Ardila, Colombia

Day 4

1 Circles w_1 and w_2 with centres O_1 and O_2 are externally tangent at point D and internally tangent to a circle w at points E and F respectively. Line t is the common tangent of w_1 and

w_2 at D. Let AB be the diameter of w perpendicular to t, so that A, E, O_1 are on the same side of t. Prove that lines AO_1, BO_2, EF and t are concurrent.

2 Find all integer solutions (x, y) of the equation $y^2 = x^3 - p^2x$, where p is a prime such that $p \equiv 3 \mod 4$.

3 Find all function(s) $f : \mathbb{R} \to \mathbb{R}$ satisfying the equation

$$f(x+y) + f(x)f(y) = (1+y)f(x) + (1+x)f(y) + f(xy);$$

For all $x, y \in \mathbb{R}$.