India International Mathematical Olympiad Training Camp 2009

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by ith_power

1 Let ABC be a triangle with $\angle A = 60^\circ$.Prove that if T is point of contact of Incircle And Nine-Point Circle, Then $AT = r$, r being inradius.

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2 Let us consider a simle graph with vertex set V. All ordered pair (a, b) of integers with $gcd(a, b) = 1$, are elements of V. (a, b) is connected to $(a, b + kab)$ by an edge and to $(a + kab, b)$ by another edge for all integer k.
Prove that for all $(a, b) \in V$, there exists a path fromm $(1, 1)$ to (a, b).

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3 Let a, b be two distinct odd natural numbers.Define a Sequence $< a_n >_{n\geq 0}$ like following: $a_1 = a$
$a_2 = b$
$a_n =$ largest odd divisor of $(a_{n-1} + a_{n-2})$.
Prove that there exists a natural number N such that $a_n = gcd(a, b) \forall n \geq N$.

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4 Let γ be circumcircle of $\triangle ABC$.Let R_a be radius of circle touching AB, AC&γ internally.Define R_b, R_c similarly.
Prove That $\frac{1}{aR_a} + \frac{1}{bR_b} + \frac{1}{cR_c} = \frac{s^2}{rabc}$.

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5 Let $f(x)$and $g(y)$ be two monic polynomials of degree=n having complex coefficients.
We know that there exist complex numbers $a_i, b_i, c_i \forall 1 \leq i \leq n$, such that $f(x) - g(y) = \prod_{i=1}^{n} (a_i x + b_i y + c_i)$.
Prove that there exists $a, b, c \in \mathbb{C}$ such that $f(x) = (x + a)^n + c$ and $g(y) = (y + b)^n + c$.

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6 Prove The Following identity: $\sum_{j=0}^{n} \left(\binom{3n+2-j}{j} 2^j - \binom{3n+1-j}{j-1} 2^{j-1} \right) = 2^{3n}$.
The Second term on left hand side is to be regarded zero for j=0.

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7 Let P be any point in the interior of a $\triangle ABC$.Prove That $\frac{PA}{a} + \frac{PB}{b} + \frac{PC}{c} \geq \sqrt{3}$.

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8 Let n be a natural number ≥ 2 which divides $3^n + 4^n$.Prove That $7 \mid n$.

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9 Let $f(x) = \sum_{k=1}^{n} a_k x^k$ and $g(x) = \sum_{k=1}^{n} \frac{a_k x^k}{2^k - 1}$ be two polynomials with real coefficients.
Let g(x) have $0, 2^{n+1}$ as two of its roots. Prove That $f(x)$ has a positive root less than 2^n.

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10 For a certain triangle all of its altitudes are integers whose sum is less than 20. If its Inradius is also an integer Find all possible values of area of the triangle.

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11 Find all integers $n \geq 2$ with the following property:

There exists three distinct primes p, q, r such that
whenever $a_1, a_2, a_3, \cdots, a_n$ are n distinct positive integers with the property that at least one of p, q, r divides $a_j - a_k \; \forall 1 \le j \le k \le n$,
one of p, q, r divides all of these differences.

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12 Let G be a simple graph with vertex set $V = \{0, 1, 2, 3, \cdots, n+1\}$.$jandj+1$ are connected by an edge for $0 \le j \le n$. Let A be a subset of V and $G(A)$ be the induced subgraph associated with A. Let $O(G(A))$ be number of components of $G(A)$ having an odd number of vertices.
Let $T(p, r) = \{A \subset V \mid 0.n+1 \notin A, |A| = p, O(G(A)) = 2r\}$ for $r \le p \le 2r$.
Prove That $|T(p, r)| = \binom{n-r}{p-r}\binom{n-p+1}{2r-p}$.
