**India International Mathematical Olympiad Training Camp 2010**

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by Akashnil, Goutham

**1** Let $ABC$ be a triangle in which $BC < AC$. Let $M$ be the mid-point of $AB$, $AP$ be the altitude from $A$ on $BC$, and $BQ$ be the altitude from $B$ on to $AC$. Suppose that $QP$ produced meets $AB$ (extended) at $T$. If $H$ is the orthocenter of $ABC$, prove that $TH$ is perpendicular to $CM$.

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**2** Two polynomials $P(x)=x^4+ax^3+bx^2+cx+d$ and $Q(x)=x^2+px+q$ have real coefficients, and $I$ is an interval on the real line of length greater than $2$. Suppose $P(x)$ and $Q(x)$ take negative values on $I$, and they take non-negative values outside $I$. Prove that there exists a real number $x_0$ such that $P(x_0)<Q(x_0)$.

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**3** For any integer $n\ge 2$, let $N(n)$ be the maximum number of triples $(a_j,b_j,c_j), j=1,2,3,\cdots,N(n)$, consisting of non-negative integers $a_j,b_j,c_j$ (not necessarily distinct) such that the following two conditions are satisfied:

(a) $a_j+b_j+c_j=n$, for all $j=1,2,3,\cdots N(n)$;
(b) $j\neq k$, then $a_j\neq a_k$, $b_j\neq b_k$ and $c_j\neq c_k$.

Determine $N(n)$ for all $n\ge 2$.

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**4** Let $a,b,c$ be positive real numbers such that $ab+bc+ca\le 3abc$. Prove that

$$\sqrt{\frac{a^2+b^2}{a+b}}+\sqrt{\frac{b^2+c^2}{b+c}}+\sqrt{\frac{c^2+a^2}{c+a}}+3\le\sqrt{2}(\sqrt{a+b}+\sqrt{b+c}+\sqrt{c+a})$$

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**5** Given an integer $k>1$, show that there exist an integer an $n>1$ and distinct positive integers $a_1,a_2,\cdots a_n$, all greater than $1$, such that the sums $\sum_{j=1}^n a_j$ and $\sum_{j=1}^n \phi(a_j)$ are both $k$-th powers of some integers.
(Here $\phi(m)$ denotes the number of positive integers less than $m$ and relatively prime to $m$.)

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**6** Let $n\ge 2$ be a given integer. Show that the number of strings of length $n$ consisting of $0'$s and $1'$s such that there are equal number of $00$ and $11$ blocks in each string is equal to

$$2\binom{n-2}{\left\lfloor\frac{n-2}{2}\right\rfloor}$$

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**7** Let $ABCD$ be a cyclic quadrilaterla and let $E$ be the point of intersection of its diagonals $AC$ and $BD$. Suppose $AD$ and $BC$ meet in $F$. Let the midpoints of $AB$ and $CD$ be $G$ and $H$ respectively. If $\Gamma$ is the circumcircle of triangle $EGH$, prove that $FE$ is tangent to $\Gamma$.

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**8** Call a positive integer **good** if either $N = 1$ or $N$ can be written as product of *even* number of prime numbers, not necessarily distinct.
Let $P(x) = (x - a)(x - b),$ where $a, b$ are positive integers.

(a) Show that there exist distinct positive integers $a, b$ such that $P(1), P(2), \cdots, P(2010)$ are all good numbers.
(b) Suppose $a, b$ are such that $P(n)$ is a good number for all positive integers $n$. Prove that $a = b$.

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**9** Let $A = (a_{jk})$ be a $10 \times 10$ array of positive real numbers such that the sum of numbers in row as well as in each column is $1$.
Show that there exists $j < k$ and $l < m$ such that

$$a_{jl}a_{km} + a_{jm}a_{kl} \geq \frac{1}{50}$$

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**10** Let $ABC$ be a triangle. Let $\Omega$ be the brocard point. Prove that $\left(\frac{A\Omega}{BC}\right)^2 + \left(\frac{B\Omega}{AC}\right)^2 + \left(\frac{C\Omega}{AB}\right)^2 \geq 1$

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**11** Find all functions $f : \mathbb{R} \longrightarrow \mathbb{R}$ such that $f(x + y) + xy = f(x)f(y)$ for all reals $x, y$

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**12** Prove that there are infinitely many positive integers $m$ for which there exists consecutive odd positive integers $p_m < q_m$ such that $p_m^2 + p_m q_m + q_m^2$ and $p_m^2 + m \cdot p_m q_m + q_m^2$ are both perfect squares. If $m_1, m_2$ are two positive integers satisfying this condition, then we have $p_{m_1} \neq p_{m_2}$

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