India International Mathematical Olympiad Training Camp 2011

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Day 1

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1 Let ABC be a triangle each of whose angles is greater than 30°. Suppose a circle centered with P cuts segments BC in $T,Q;CA$ in K,L and AB in M,N such that they are on a circle in counterclockwise direction in that order.Suppose further PQK,PLM,PNT are equilateral. Prove that:

$a)$ The radius of the circle is $\frac{2abc}{a^2+b^2+c^2+4\sqrt{3}S}$ where S is area.

$b)a\cdot AP=b\cdot BP=c\cdot PC.$

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2 Let the real numbers a,b,c,d satisfy the relations $a+b+c+d=6$ and $a^2+b^2+c^2+d^2=12.$ Prove that

$$36 \leq 4 \left(a^3+b^3+c^3+d^3\right) - \left(a^4+b^4+c^4+d^4\right) \leq 48.$$

Proposed by Nazar Serdyuk, Ukraine

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3 A set of n distinct integer weights $w_1,w_2,\ldots ,w_n$ is said to be *balanced* if after removing any one of weights, the remaining $(n-1)$ weights can be split into two subcollections (not necessarily with equal size)with equal sum.

$a)$ Prove that if there exist *balanced* sets of sizes k,j then also a *balanced* set of size $k+j-1$.
$b)$ Prove that for all *odd* $n\geq 7$ there exist a *balanced* set of size n.

Day 2

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1 Find all positive integer n satisfying the conditions

$a)n^2=(a+1)^3-a^3$

$b)2n+119$ is a perfect square.

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2 Suppose $a_1,\ldots,a_n$ are non-integral real numbers for $n\geq 2$ such that ${a_1}^k+\ldots+{a_n}^k$ is an integer for all integers $1\leq k\leq n$. Prove that none of $a_1,\ldots,a_n$ is rational.

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3 Let T be a non-empty finite subset of positive integers ≥ 1. A subset S of T is called **good** if for every integer $t\in T$ there exists an s in S such that $gcd(t,s) > 1$. Let

$$A = (X,Y) \mid X \subseteq T, Y \subseteq T, gcd(x,y) = 1 \text{for all} x \in X, y \in Y$$

Prove that : $a)$ If X_0 is not **good** then the number of pairs (X_0, Y) in A is **even**. $b)$ the number of good subsets of T is **odd**.

Day 3

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1 Let $ABCDE$ be a convex pentagon such that $BC \parallel AE$, $AB = BC + AE$, and $\angle ABC = \angle CDE$. Let M be the midpoint of CE, and let O be the circumcenter of triangle BCD. Given that $\angle DMO = 90^\circ$, prove that $2\angle BDA = \angle CDE$.

Proposed by Nazar Serdyuk, Ukraine

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2 Prove that for no integer n is $n^7 + 7$ a perfect square.

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3 Consider a $n \times n$ square grid which is divided into n^2 unit squares(think of a chess-board). The set of all unit squares intersecting the main diagonal of the square or lying under it is called an n-staircase. Find the number of ways in which an n-stair case can be partitioned into several rectangles, with sides along the grid lines, having mutually distinct areas.

Day 4

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1 Let ABC be an acute-angled triangle. Let AD, BE, CF be internal bisectors with D, E, F on BC, CA, AB respectively. Prove that

$$\frac{EF}{BC} + \frac{FD}{CA} + \frac{DE}{AB} \geq 1 + \frac{r}{R}$$

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2 Find all pairs (m, n) of nonnegative integers for which

$$m^2 + 2 \cdot 3^n = m\left(2^{n+1} - 1\right).$$

Proposed by Angelo Di Pasquale, Australia

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3 Let $\{a_0, a_1, \ldots\}$ and $\{b_0, b_1, \ldots\}$ be two infinite sequences of integers such that

$$(a_n - a_{n-1})(a_n - a_{n-2}) + (b_n - b_{n-1})(b_n - b_{n-2}) = 0$$

for all integers $n \geq 2$. Prove that there exists a positive integer k such that

$$a_{k+2011} = a_{k+2011^{2011}}.$$
