India International Mathematical Olympiad Training Camp 2012

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–	Practice Tests
Day 1	
1	Let ABC be an isosceles triangle with $AB = AC$. Let D be a point on the segment BC such that $BD = 2DC$. Let P be a point on the segment AD such that $\angle BAC = \angle BPD$. Prove that $\angle BAC = 2\angle DPC$.
2	Let $a \geq b$ and $c \geq d$ be real numbers. Prove that the equation $$(x+a)(x+d)+(x+b)(x+c)=0$$ has real roots.
3	How many 6-tuples (a,b,c,d,e,f) of natural numbers are there for which $a>b>c>d>e>f$ and $a+f=b+e=c+d=30$ are simultaneously true?
Day 2	
1	Let $ABCD$ be a trapezium with $AB \parallel CD$. Let P be a point on AC such that C is between A and P; and let X, Y be the midpoints of AB, CD respectively. Let PX intersect BC in N and PY intersect AD in M. Prove that $MN \parallel AB$.
2	Let $0<x<y<z<p$ be integers where p is a prime. Prove that the following statements are equivalent: $(a) x^3 \equiv y^3 \pmod p$ and $x^3 \equiv z^3 \pmod p$ $(b) y^2 \equiv zx \pmod p$ and $z^2 \equiv xy \pmod p$
3	Let $f : \mathbb{R} \longrightarrow \mathbb{R}$ be a function such that $f(x+y+xy) = f(x)+f(y)+f(xy)$ for all $x, y \in \mathbb{R}$. Prove that f satisfies $f(x+y) = f(x)+f(y)$ for all $x, y \in \mathbb{R}$.
–	Team Selection Tests
Day 1	
1	The cirumcentre of the cyclic quadrilateral $ABCD$ is O. The second intersection point of the circles ABO and CDO, other than O, is P, which lies in the interior of the triangle DAO. Choose a point Q on the extension of OP beyond P, and a point R on the extension of OP beyond O. Prove that $\angle QAP = \angle OBR$ if and only if $\angle PDQ = \angle RCO$.

2 Let $P(z)=a_nz^n+a_{n-1}z^{n-1}+\ldots+a_mz^m$ be a polynomial with complex coefficients such that $a_m\neq 0, a_n\neq 0$ and $n>m$. Prove that

$$\text{max}_{|z|=1}\{|P(z)|\}\geq\sqrt{2|a_ma_n|+\sum_{k=m}^{n}|a_k|^2}$$

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3 Determine the greatest positive integer k that satisfies the following property: The set of positive integers can be partitioned into k subsets $A_1, A_2, \ldots, A_k$ such that for all integers $n \geq 15$ and all $i \in \{1, 2, \ldots, k\}$ there exist two distinct elements of A_i whose sum is n.

Proposed by Igor Voronovich, Belarus

Day 2

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1 Determine all sequences $(x_1,x_2,\ldots,x_{2011})$ of positive integers, such that for every positive integer n there exists an integer a with

$$\sum^{2011}_{j=1} j x^n_j = a^{n+1} + 1$$

Proposed by Warut Suksompong, Thailand

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2 Show that there exist infinitely many pairs (a,b) of positive integers with the property that $a+b$ divides $ab+1$, $a-b$ divides $ab-1$, $b>1$ and $a>b\sqrt{3}-1$

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3 Suppose that 1000 students are standing in a circle. Prove that there exists an integer k with $100 \leq k \leq 300$ such that in this circle there exists a contiguous group of $2k$ students, for which the first half contains the same number of girls as the second half.

Proposed by Gerhard Wginger, Austria

Day 3

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1 Let ABC be a triangle with $AB=AC$ and let D be the midpoint of AC. The angle bisector of $\angle BAC$ intersects the circle through D,B and C at the point E inside the triangle ABC. The line BD intersects the circle through A,E and B in two points B and F. The lines AF and BE meet at a point I, and the lines CI and BD meet at a point K. Show that I is the incentre of triangle KAB.

Proposed by Jan Vonk, Belgium and Hojoo Lee, South Korea

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2 Let S be a nonempty set of primes satisfying the property that for each proper subset P of S, all the prime factors of the number $\left(\prod_{p\in P} p\right) - 1$ are also in S. Determine all possible such sets S.

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3 In a $2 \times n$ array we have positive reals s.t. the sum of the numbers in each of the n columns is 1. Show that we can select a number in each column s.t. the sum of the selected numbers in each row is at most $\frac{n+1}{4}$.

Day 4

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1 A quadrilateral $ABCD$ without parallel sides is circumscribed around a circle with centre O. Prove that O is a point of intersection of middle lines of quadrilateral $ABCD$ (i.e. barycentre of points $A,\ B,\ C,\ D$) iff $OA \cdot OC = OB \cdot OD$.

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2 Find the least positive integer that cannot be represented as $\frac{2^a-2^b}{2^c-2^d}$ for some positive integers a, b, c, d.

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3 Let $\mathbb{R}^+$ denote the set of all positive real numbers. Find all functions $f : \mathbb{R}^+ \longrightarrow \mathbb{R}$ satisfying

$$f(x)+f(y) \le \frac{f(x+y)}{2}, \frac{f(x)}{x}+\frac{f(y)}{y} \ge \frac{f(x+y)}{x+y},$$

for all $x, y \in \mathbb{R}^+$.
