India International Mathematical Olympiad Training Camp 2013

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by semisimplicity, Goutham, lyukhson

– Practice Test

Day 1 May 5th

1 For a prime p, a natural number n and an integer a, we let $S_n(a,p)$ denote the exponent of p in the prime factorisation of $a^{p^n}-1$. For example, $S_1(4,3)=2$ and $S_2(6,2)=0$. Find all pairs (n,p) such that $S_n(2013,p)=100$.

2 Let $ABCD$ by a cyclic quadrilateral with circumcenter O. Let P be the point of intersection of the diagonals AC and BD, and K, L, M, N the circumcenters of triangles AOP, BOP, COP, DOP, respectively. Prove that $KL = MN$.

3 We define an operation $\oplus$ on the set $\{0,1\}$ by

$$0\oplus 0=0\,,0\oplus 1=1\,,1\oplus 0=1\,,1\oplus 1=0\,.$$

For two natural numbers a and b, which are written in base 2 as $a=(a_1a_2\ldots a_k)_2$ and $b=(b_1b_2\ldots b_k)_2$ (possibly with leading 0's), we define $a\oplus b=c$ where c written in base 2 is $(c_1c_2\ldots c_k)_2$ with $c_i=a_i\oplus b_i$, for $1\le i\le k$. For example, we have $7\oplus 3=4$ since $7=(111)_2$ and $3=(011)_2$.

For a natural number n, let $f(n)=n\oplus [n/2]$, where $[x]$ denotes the largest integer less than or equal to x. Prove that f is a bijection on the set of natural numbers.

Day 2 May 10th

1 Let a,b,c be positive real numbers such that $a+b+c=1$. If n is a positive integer then prove that

$$\frac{(3a)^n}{(b+1)(c+1)}+\frac{(3b)^n}{(c+1)(a+1)}+\frac{(3c)^n}{(a+1)(b+1)}\ge\frac{27}{16}\,.$$

2 In a triangle ABC with $B=90^\circ$, D is a point on the segment BC such that the inradii of triangles ABD and ADC are equal. If $\widehat{ADB}=\varphi$ then prove that $\tan^2(\varphi/2)=\tan(C/2)$.

3 A marker is placed at the origin of an integer lattice. Calvin and Hobbes play the following game. Calvin starts the game and each of them takes turns alternatively. At each turn, one can choose two (not necessarily distinct) integers a,b, neither of which was chosen earlier by any player and move the marker by a units in the horizontal direction and b units in the vertical direction. Hobbes wins if the marker is back at the origin any time after the first turn. Prove or disprove that Calvin can prevent Hobbes from winning.

Note: A move in the horizontal direction by a positive quantity will be towards the right, and by a negative quantity will be towards the left (and similar directions in the vertical case as well).

– Team Selection Test

Day 1 May 15th

1 Let $n \geq 2$ be an integer. There are n beads numbered $1, 2, \ldots, n$. Two necklaces made out of some of these beads are considered the same if we can get one by rotating the other (with no flipping allowed). For example, with $n \geq 5$, the necklace with four beads $1, 5, 3, 2$ in the clockwise order is same as the one with $5, 3, 2, 1$ in the clockwise order, but is different from the one with $1, 2, 3, 5$ in the clockwise order.

We denote by $D_0(n)$ (respectively $D_1(n)$) the number of ways in which we can use all the beads to make an even number (resp. an odd number) of necklaces each of length at least 3. Prove that $n-1$ divides $D_1(n) - D_0(n)$.

2 In a triangle ABC, with $\widehat{A} > 90^\circ$, let O and H denote its circumcenter and orthocenter, respectively. Let K be the reflection of H with respect to A. Prove that K, O and C are collinear if and only if $\widehat{A} - \widehat{B} = 90^\circ$.

3 For a positive integer n, a cubic polynomial $p(x)$ is said to be [i]n-good[/i] if there exist n distinct integers $a_1, a_2, \ldots, a_n$ such that all the roots of the polynomial $p(x) + a_i = 0$ are integers for $1 \leq i \leq n$. Given a positive integer n prove that there exists an n-good cubic polynomial.

Day 2 May 16th

1 Find all functions f from the set of real numbers to itself satisfying

$$f(x(1+y)) = f(x)(1+f(y))$$

for all real numbers x, y.

2 An integer a is called friendly if the equation $(m^2+n)(n^2+m) = a(m-n)^3$ has a solution over the positive integers.
a) Prove that there are at least 500 friendly integers in the set $\{1, 2, \ldots, 2012\}$.
b) Decide whether $a = 2$ is friendly.

3 Players A and B play a game with $N \geq 2012$ coins and 2012 boxes arranged around a circle. Initially A distributes the coins among the boxes so that there is at least 1 coin in each box.

Then the two of them make moves in the order $B, A, B, A, \dots$ by the following rules:
(a) On every move of his B passes 1 coin from every box to an adjacent box.
(b) On every move of hers A chooses several coins that were *not* involved in B's previous move and are in different boxes. She passes every coin to an adjacent box.
Player A's goal is to ensure at least 1 coin in each box after every move of hers, regardless of how B plays and how many moves are made. Find the least N that enables her to succeed.

Day 3 May 22nd

1 For a positive integer n, a *sum-friendly odd partition* of n is a sequence $(a_1, a_2, \dots, a_k)$ of odd positive integers with $a_1 \le a_2 \le \cdots \le a_k$ and $a_1 + a_2 + \cdots + a_k = n$ such that for all positive integers $m \le n$, m can be **uniquely** written as a subsum $m = a_{i_1} + a_{i_2} + \cdots + a_{i_r}$. (Two subsums $a_{i_1} + a_{i_2} + \cdots + a_{i_r}$ and $a_{j_1} + a_{j_2} + \cdots + a_{j_s}$ with $i_1 < i_2 < \cdots < i_r$ and $j_1 < j_2 < \cdots < j_s$ are considered the same if $r = s$ and $a_{i_l} = a_{j_l}$ for $1 \le l \le r$.) For example, $(1, 1, 3, 3)$ is a sum-friendly odd partition of 8. Find the number of sum-friendly odd partitions of 9999.

2 In a triangle ABC, let I denote its incenter. Points D, E, F are chosen on the segments BC, CA, AB, respectively, such that $BD + BF = AC$ and $CD + CE = AB$. The circumcircles of triangles AEF, BFD, CDE intersect lines AI, BI, CI, respectively, at points K, L, M (different from A, B, C), respectively. Prove that K, L, M, I are concyclic.

3 Let $h \ge 3$ be an integer and X the set of all positive integers that are greater than or equal to $2h$. Let S be a nonempty subset of X such that the following two conditions hold:

-if $a + b \in S$ with $a \ge h, b \ge h$, then $ab \in S$;

-if $ab \in S$ with $a \ge h, b \ge h$, then $a + b \in S$.
Prove that $S = X$.

Day 4 May 23rd

1 A positive integer a is called a *double number* if it has an even number of digits (in base 10) and its base 10 representation has the form $a = a_1a_2 \cdots a_k a_1 a_2 \cdots a_k$ with $0 \le a_i \le 9$ for $1 \le i \le k$, and $a_1 \ne 0$. For example, 283283 is a double number. Determine whether or not there are infinitely many double numbers a such that $a + 1$ is a square and $a + 1$ is not a power of 10.

2 Let $n \ge 2$ be an integer and $f_1(x), f_2(x), \dots, f_n(x)$ a sequence of polynomials with integer coefficients. One is allowed to make moves $M_1, M_2, \dots$ as follows: in the k-th move M_k one chooses an element $f(x)$ of the sequence with degree of f at least 2 and replaces it with $(f(x) - f(k))/(x - k)$. The process stops when all the elements of the sequence are of degree 1. If $f_1(x) = f_2(x) = \cdots = f_n(x) = x^n + 1$, determine whether or not it is possible to make appropriate moves such that the process stops with a sequence of n identical polynomials of degree 1.

3 In a triangle ABC, with $AB \neq BC$, E is a point on the line AC such that BE is perpendicular to AC. A circle passing through A and touching the line BE at a point $P \neq B$ intersects the line AB for the second time at X. Let Q be a point on the line PB different from P such that $BQ = BP$. Let Y be the point of intersection of the lines CP and AQ. Prove that the points C, X, Y, A are concyclic if and only if CX is perpendicular to AB.