India International Mathematical Olympiad Training Camp 2014

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–	Practice Test 1
–	April 27th
1	Let p be an odd prime and r an odd natural number.Show that $pr+1$ does not divide p^p-1
2	Let a,b be positive real numbers.Prove that $(1+a)^{8}+(1+b)^{8}\geq 128ab(a+b)^{2}$.
3	In a triangle ABC, points X and Y are on BC and CA respectively such that $CX=CY$,AX is not perpendicular to BC and BY is not perpendicular to CA.Let Γ be the circle with C as centre and CX as its radius.Find the angles of triangle ABC given that the orthocentres of triangles AXB and AYB lie on Γ.
–	Practice Test 2
–	May 2nd
1	Let x and y be rational numbers, such that $x^{5}+y^{5}=2x^{2}y^{2}$. Prove that $1-xy$ is the square of a rational number.
2	Let n be a natural number.A triangulation of a convex n-gon is a division of the polygon into $n-2$ triangles by drawing $n-3$ diagonals no two of which intersect at an interior point of the polygon.Let $f(n)$ denote the number of triangulations of a regular n-gon such that each of the triangles formed is isosceles.Determine $f(n)$ in terms of n.
3	For integers a,b we define $f((a,b))=(2a,b-a)$ if $a<b$ and $f((a,b))=(a-b,2b)$ if $a\geq b$. Given a natural number $n>1$ show that there exist natural numbers m,k with $m<n$ such that $f^{k}((n,m))=(m,n)$,where $f^{k}(x)=f(f(f(...f(x))))$,$f$ being composed with itself k times.
–	TST 1
–	May 7th
1	Find all polynomials $f(x)$ with integer coefficients such that $f(n)$ and $f(2^{n})$ are co-prime for all natural numbers n.
2	Let n be an positive integer. Find the smallest integer k with the following property; Given any real numbers $a_1 , \cdots , a_d$ such that $a_1 + a_2 + \cdots + a_d = n$ and $0 \le a_i \le 1$ for $i=1,2,\cdots ,d$, it

is possible to partition these numbers into k groups (some of which may be empty) such that the sum of the numbers in each group is at most 1.

3 Starting with the triple $(1007\sqrt{2}, 2014\sqrt{2}, 1007\sqrt{14})$, define a sequence of triples (x_n, y_n, z_n) by
$x_{n+1} = \sqrt{x_n(y_n + z_n - x_n)}$
$y_{n+1} = \sqrt{y_n(z_n + x_n - y_n)}$ $z_{n+1} = \sqrt{z_n(x_n + y_n - z_n)}$
for $n \geq 0$.Show that each of the sequences $\langle x_n\rangle_{n\geq 0}, \langle y_n\rangle_{n\geq 0}, \langle z_n\rangle_{n\geq 0}$ converges to a limit and find these limits.

– TST 2

– May 8th

1 In a triangle ABC, let I be its incenter; Q the point at which the incircle touches the line AC; E the midpoint of AC and K the orthocenter of triangle BIC. Prove that the line KQ is perpendicular to the line IE.

2 For $j = 1, 2, 3$ let x_j, y_j be non-zero real numbers, and let $v_j = x_j + y_j$.Suppose that the following statements hold:

$x_1x_2x_3 = -y_1y_2y_3$

$x_1^2 + x_2^2 + x_3^2 = y_1^2 + y_2^2 + y_3^2$

v_1, v_2, v_3 satisfy triangle inequality

v_1^2, v_2^2, v_3^2 also satisfy triangle inequality.

Prove that exactly one of $x_1, x_2, x_3, y_1, y_2, y_3$ is negative.

3 Let r be a positive integer, and let $a_0, a_1, \cdots$ be an infinite sequence of real numbers. Assume that for all nonnegative integers m and s there exists a positive integer $n \in [m+1, m+r]$ such that

$$a_m + a_{m+1} + \cdots + a_{m+s} = a_n + a_{n+1} + \cdots + a_{n+s}$$

Prove that the sequence is periodic, i.e. there exists some $p \geq 1$ such that $a_{n+p} = a_n$ for all $n \geq 0$.

– TST 3

– May 14th

1 In a triangle ABC, with $AB \neq AC$ and $A \neq 60^0, 120^0$, D is a point on line AC different from C. Suppose that the circumcentres and orthocentres of triangles ABC and ABD lie on a circle. Prove that $\angle ABD = \angle ACB$.

2 Determine whether there exists an infinite sequence of nonzero digits $a_1, a_2, a_3, \cdots$ and a positive integer N such that for every integer $k > N$, the number $\overline{a_k a_{k-1} \cdots a_1}$ is a perfect square.

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3 In how many ways rooks can be placed on a 8 by 8 chess board such that every row and every column has at least one rook?
(Any number of rooks are available,each square can have at most one rook and there is no relation of attacking between them)

– TST 4

– May 15th

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1 Prove that in any set of 2000 distinct real numbers there exist two pairs $a > b$ and $c > d$ with $a \neq c$ or $b \neq d$, such that

$$\left| \frac{a-b}{c-d} - 1 \right| < \frac{1}{100000}.$$

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2 Find all positive integers x and y such that $x^{x+y} = y^{3x}$.

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3 Let ABC be a triangle with $\angle B > \angle C$. Let P and Q be two different points on line AC such that $\angle PBA = \angle QBA = \angle ACB$ and A is located between P and C. Suppose that there exists an interior point D of segment BQ for which $PD = PB$. Let the ray AD intersect the circle ABC at $R \neq A$. Prove that $QB = QR$.
