Canada National Olympiad 1973

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by chess64, A M Y, ch1n353ch3s54a1l, boxedexe, Kurt Gdel, scorpius119

1 (i) Solve the simultaneous inequalities, $x < \frac{1}{4x}$ and $x < 0$; i.e. find a single inequality equivalent to the two simultaneous inequalities.

(ii) What is the greatest integer that satisfies both inequalities $4x + 13 < 0$ and $x^2 + 3x > 16$.

(iii) Give a rational number between $11/24$ and $6/13$.

(iv) Express 100000 as a product of two integers neither of which is an integral multiple of 10.

(v) Without the use of logarithm tables evaluate

$$\frac{1}{\log_2 36} + \frac{1}{\log_3 36}.$$

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2 Find all real numbers that satisfy the equation $|x+3| - |x-1| = x+1$. (Note: $|a| = a$ if $a \geq 0$; $|a| = -a$ if $a < 0$.)

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3 Prove that if p and $p+2$ are prime integers greater than 3, then 6 is a factor of $p+1$.

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4 The figure shows a (convex) polygon with nine vertices. The six diagonals which have been drawn dissect the polygon into the seven triangles: $P_0P_1P_3$, $P_0P_3P_6$, $P_0P_6P_7$, $P_0P_7P_8$, $P_1P_2P_3$, $P_3P_4P_6$, $P_4P_5P_6$. In how many ways can these triangles be labeled with the names $\triangle_1$, $\triangle_2$, $\triangle_3$, $\triangle_4$, $\triangle_5$, $\triangle_6$, $\triangle_7$ so that P_i is a vertex of triangle $\triangle_i$ for $i = 1, 2, 3, 4, 5, 6, 7$? Justify your answer.

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5 For every positive integer n, let

$$h(n) = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}.$$

For example, $h(1) = 1$, $h(2) = 1 + \frac{1}{2}$, $h(3) = 1 + \frac{1}{2} + \frac{1}{3}$. Prove that for $n = 2, 3, 4, \ldots$

$$n + h(1) + h(2) + h(3) + \cdots + h(n-1) = nh(n).$$

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6 If A and B are fixed points on a given circle not collinear with centre O of the circle, and if XY is a variable diameter, find the locus of P (the intersection of the line through A and X and the line through B and Y).

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7 Observe that

$$\frac{1}{1}=\frac{1}{2}+\frac{1}{2};\quad \frac{1}{2}=\frac{1}{3}+\frac{1}{6};\quad \frac{1}{3}=\frac{1}{4}+\frac{1}{12};\quad \frac{1}{4}=\frac{1}{5}+\frac{1}{20}.$$

State a general law suggested by these examples, and prove it.

Prove that for any integer n greater than 1 there exist positive integers i and j such that

$$\frac{1}{n}=\frac{1}{i(i+1)}+\frac{1}{(i+1)(i+2)}+\frac{1}{(i+2)(i+3)}+\cdots+\frac{1}{j(j+1)}.$$

It seems that this is a two-part problem.
