

Canada National Olympiad 1975

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by Silverfalcon

1 Simplify

$$\left(\frac{1 \cdot 2 \cdot 4 + 2 \cdot 4 \cdot 8 + \dots + n \cdot 2n \cdot 4n}{1 \cdot 3 \cdot 9 + 2 \cdot 6 \cdot 18 + \dots + n \cdot 3n \cdot 9n} \right)^{\frac{1}{3}}$$

2 A sequence of numbers a_1, a_2, a_3, \dots satisfies

(i) $a_1 = \frac{1}{2}$

(ii) $a_1 + a_2 + \dots + a_n = n^2 a_n$ ($n \geq 1$)

Determine the value of a_n ($n \geq 1$).

3 For each real number r , $[r]$ denotes the largest integer less than or equal to r , e.g. $[6] = 6$, $[\pi] = 3$, $[-1.5] = -2$. Indicate on the (x, y) -plane the set of all points (x, y) for which $[x]^2 + [y]^2 = 4$.

4 For a positive number such as 3.27, 3 is referred to as the integral part of the number and .27 as the decimal part. Find a positive number such that its decimal part, its integral part, and the number itself form a geometric progression.

5 A, B, C, D are four "consecutive" points on the circumference of a circle and P, Q, R, S are points on the circumference which are respectively the midpoints of the arcs AB, BC, CD, DA . Prove that PR is perpendicular to QS .

6 (i) 15 chairs are equally placed around a circular table on which are name cards for 15 guests. The guests fail to notice these cards until after they have sat down, and it turns out that no one is sitting in the correct seat. Prove that the table can be rotated so that at least two of the guests are simultaneously correctly seated.

(ii) Give an example of an arrangement in which just one of the 15 guests is correctly seated and for which no rotation correctly places more than one person.

7 A function $f(x)$ is *periodic* if there is a positive number p such that $f(x+p) = f(x)$ for all x . For example, $\sin x$ is periodic with period 2π . Is the function $\sin(x^2)$ periodic? Prove your assertion.

8 Let k be a positive integer. Find all polynomials

$$P(x) = a_0 + a_1x + \dots + a_nx^n,$$

where the a_i are real, which satisfy the equation

$$P(P(x)) = \{P(x)\}^k$$
