Canada National Olympiad 1982

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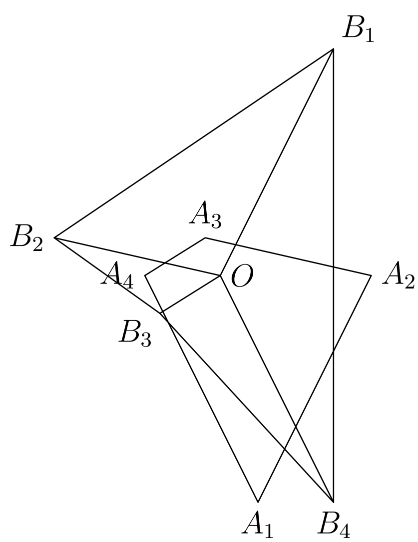
1 In the diagram, OB_i is parallel and equal in length to A_iA_{i+1} for $i = 1$, 2, 3, and 4 ($A_5 = A_1$). Show that the area of $B_1B_2B_3B_4$ is twice that of $A_1A_2A_3A_4$.



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2 If a, b and c are the roots of the equation $x^3 - x^2 - x - 1 = 0$,
(i) show that a, b and c are distinct:
(ii) show that

$$\frac{a^{1982}-b^{1982}}{a-b}+\frac{b^{1982}-c^{1982}}{b-c}+\frac{c^{1982}-a^{1982}}{c-a}$$

is an integer.

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3 Let $\mathbb{R}^n$ be the n-dimensional Euclidean space. Determine the smallest number $g(n)$ of a points of a set in $\mathbb{R}^n$ such that every point in $\mathbb{R}^n$ is an irrational distance from at least one point in that set.

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4 Let p be a permutation of the set $S_n = \{1, 2, \ldots, n\}$. An element $j \in S_n$ is called a fixed point of p if $p(j) = j$. Let f_n be the number of permutations having no fixed points, and g_n be the number with exactly one fixed point. Show that $|f_n - g_n| = 1$.

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5 The altitudes of a tetrahedron $ABCD$ are extended externally to points A', B', C', and D', where

$AA' = k/h_a$, $BB' = k/h_b$, $CC' = k/h_c$, and $DD' = k/h_d$. Here, k is a constant and h_a denotes the length of the altitude of $ABCD$ from vertex A, etc. Prove that the centroid of tetrahedron $A'B'C'D'$ coincides with the centroid of $ABCD$.