Canada National Olympiad 1984

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1 Prove that the sum of the squares of 1984 consecutive positive integers cannot be the square of an integer.

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2 Alice and Bob are in a hardware store. The store sells coloured sleeves that fit over keys to distinguish them. The following conversation takes place:
Alice: Are you going to cover your keys?
Bob: I would like to, but there are only 7 colours and I have 8 keys.
Alice: Yes, but you could always distinguish a key by noticing that the red key next to the green key was different from the red key next to the blue key.
Bob: You must be careful what you mean by "*next to*" or "*three keys over from*" since you can turn the key ring over and the keys are arranged in a circle.
Alice: Even so, you don't need 8 colours.
Problem: What is the smallest number of colours needed to distinguish n keys if all the keys are to be covered.

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3 An integer is digitally divisible if both of the following conditions are fulfilled: (a) None of its digits is zero; (b) It is divisible by the sum of its digits
e.g. 322 is digitally divisible. Show that there are infinitely many digitally divisible integers.

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4 An acute triangle has unit area. Show that there is a point inside the triangle whose distance from each of the vertices is at least $\frac{2}{\sqrt[4]{27}}$.

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5 Given any 7 real numbers, prove that there are two of them x, y such that $0 \leq \frac{x-y}{1+xy} \leq \frac{1}{\sqrt{3}}$.

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