Middle European Mathematical Olympiad 2017

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–	Individual Competition
1	Determine all functions $f:\mathbb{R}\rightarrow\mathbb{R}$ satisfying $$f(x^2+f(x)f(y))=xf(x+y)$$ for all real numbers x and y.
2	Let $n\geq 3$ be an integer. A labelling of the n vertices, the n sides and the interior of a regular n-gon by $2n+1$ distinct integers is called *memorable* if the following conditions hold: (a) Each side has a label that is the arithmetic mean of the labels of its endpoints. (b) The interior of the n-gon has a label that is the arithmetic mean of the labels of all the vertices. Determine all integers $n\geq 3$ for which there exists a memorable labelling of a regular n-gon consisting of $2n+1$ consecutive integers.
3	Let $ABCDE$ be a convex pentagon. Let P be the intersection of the lines CE and BD. Assume that $\angle PAD=\angle ACB$ and $\angle CAP=\angle EDA$. Prove that the circumcentres of the triangles ABC and ADE are collinear with P.
4	Determine the smallest possible value of $$\|2^m-181^n\|,$$ where m and n are positive integers.
–	Team Competition
1	Determine all pairs of polynomials (P,Q) with real coefficients satisfying $$P(x+Q(y))=Q(x+P(y))$$ for all real numbers x and y.
2	Determine the smallest possible real constant C such that the inequality $$\|x^3+y^3+z^3+1\|\leq C\|x^5+y^5+z^5+1\|$$ holds for all real numbers x,y,z satisfying $x+y+z=-1$.

3 There is a lamp on each cell of a 2017×2017 board. Each lamp is either on or off. A lamp is called *bad* if it has an even number of neighbours that are on. What is the smallest possible number of bad lamps on such a board?
(Two lamps are neighbours if their respective cells share a side.)

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4 Let $n \geq 3$ be an integer. A sequence $P_1, P_2, \ldots, P_n$ of distinct points in the plane is called *good* if no three of them are collinear, the polyline $P_1P_2 \ldots P_n$ is non-self-intersecting and the triangle $P_iP_{i+1}P_{i+2}$ is oriented counterclockwise for every $i = 1, 2, \ldots, n-2$.
For every integer $n \geq 3$ determine the greatest possible integer k with the following property: there exist n distinct points $A_1, A_2, \ldots, A_n$ in the plane for which there are k distinct permutations $\sigma : \{1, 2, \ldots, n\} \to \{1, 2, \ldots, n\}$ such that $A_{\sigma(1)}, A_{\sigma(2)}, \ldots, A_{\sigma(n)}$ is good.
(A polyline $P_1P_2 \ldots P_n$ consists of the segments $P_1P_2, P_2P_3, \ldots, P_{n-1}P_n$.)

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5 Let ABC be an acute-angled triangle with $AB > AC$ and circumcircle Γ. Let M be the midpoint of the shorter arc BC of Γ, and let D be the intersection of the rays AC and BM. Let $E \neq C$ be the intersection of the internal bisector of the angle ACB and the circumcircle of the triangle BDC. Let us assume that E is inside the triangle ABC and there is an intersection N of the line DE and the circle Γ such that E is the midpoint of the segment DN.
Show that N is the midpoint of the segment I_BI_C, where I_B and I_C are the excentres of ABC opposite to B and C, respectively.

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6 Let ABC be an acute-angled triangle with $AB \neq AC$, circumcentre O and circumcircle Γ. Let the tangents to Γ at B and C meet each other at D, and let the line AO intersect BC at E. Denote the midpoint of BC by M and let AM meet Γ again at $N \neq A$. Finally, let $F \neq A$ be a point on Γ such that A, M, E and F are concyclic. Prove that FN bisects the segment MD.

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7 Determine all integers $n \geq 2$ such that there exists a permutation $x_0, x_1, \ldots, x_{n-1}$ of the numbers $0, 1, \ldots, n-1$ with the property that the n numbers

$$x_0, \quad x_0 + x_1, \quad \ldots, \quad x_0 + x_1 + \ldots + x_{n-1}$$

are pairwise distinct modulo n.

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8 For an integer $n \geq 3$ we define the sequence $\alpha_1, \alpha_2, \ldots, \alpha_k$ as the sequence of exponents in the prime factorization of $n! = p_1^{\alpha_1} p_2^{\alpha_2} \ldots p_k^{\alpha_k}$, where $p_1 < p_2 < \ldots < p_k$ are primes. Determine all integers $n \geq 3$ for which $\alpha_1, \alpha_2, \ldots, \alpha_k$ is a geometric progression.
