

Canada National Olympiad 1993

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by 4everwise

- 1 Determine a triangle for which the three sides and an altitude are four consecutive integers and for which this altitude partitions the triangle into two right triangles with integer sides. Show that there is only one such triangle.
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- 2 Show that the number x is rational if and only if three distinct terms that form a geometric progression can be chosen from the sequence

$$x, x + 1, x + 2, x + 3, \dots$$

- 3 In triangle ABC , the medians to the sides \overline{AB} and \overline{AC} are perpendicular. Prove that $\cot B + \cot C \geq \frac{2}{3}$.
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- 4 A number of schools took part in a tennis tournament. No two players from the same school played against each other. Every two players from different schools played exactly one match against each other. A match between two boys or between two girls was called a *single* and that between a boy and a girl was called a *mixed single*. The total number of boys differed from the total number of girls by at most 1. The total number of singles differed from the total number of mixed singles by at most 1. At most how many schools were represented by an odd number of players?
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- 5 Let y_1, y_2, y_3, \dots be a sequence such that $y_1 = 1$ and, for $k > 0$, is defined by the relationship:

$$y_{2k} = \begin{cases} 2y_k & \text{if } k \text{ is even} \\ 2y_k + 1 & \text{if } k \text{ is odd} \end{cases}$$
$$y_{2k+1} = \begin{cases} 2y_k & \text{if } k \text{ is odd} \\ 2y_k + 1 & \text{if } k \text{ is even} \end{cases}$$

Show that the sequence takes on every positive integer value exactly once.
