Canada National Olympiad 1995

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by BigSams

1 Let $f(x) = \frac{9^x}{9^x+3}$. Evaluate $\sum_{i=1}^{1995} f\left(\frac{i}{1996}\right)$.

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2 Let $\{a, b, c\} \in \mathbb{R}^+$. Prove that $a^a b^b c^c \geq (abc)^{\frac{a+b+c}{3}}$.

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3 Define a boomerang as a quadrilateral whose opposite sides do not intersect and one of whose internal angles is greater than 180°. Let C be a convex polygon with s sides. The interior region of C is the union of q quadrilaterals, none of whose interiors overlap each other. b of these quadrilaterals are boomerangs. Show that $q \geq b + \frac{s-2}{2}$.

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4 Let n be a constant positive integer. Show that for only non-negative integers k, the Diophantine equation $\sum_{i=1}^{n} x_i^3 = y^{3k+2}$ has infinitely many solutions in the positive integers x_i, y.

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5 u is a real parameter such that $0 < u < 1$.
For $0 \leq x \leq u$, $f(x) = 0$.
For $u \leq x \leq n$, $f(x) = 1 - \left(\sqrt{ux} + \sqrt{(1-u)(1-x)}\right)^2$.
The sequence $\{u_n\}$ is define recursively as follows: $u_1 = f(1)$ and $u_n = f(u_{n-1})\ \forall n \in \mathbb{N}, n \neq 1$.
Show that there exists a positive integer k for which $u_k = 0$.
