Canada National Olympiad 1996

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by shobber

1 If α, β, and γ are the roots of $x^3 - x - 1 = 0$, compute $\frac{1+\alpha}{1-\alpha} + \frac{1+\beta}{1-\beta} + \frac{1+\gamma}{1-\gamma}$.

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2 Find all real solutions to the following system of equations. Carefully justify your answer.

$$\begin{cases} \dfrac{4x^2}{1+4x^2} = y \\ \dfrac{4y^2}{1+4y^2} = z \\ \dfrac{4z^2}{1+4z^2} = x \end{cases}$$

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3 We denote an arbitrary permutation of the integers $1, 2, \ldots, n$ by $a_1, a_2, \ldots, a_n$. Let $f(n)$ denote the number of these permutations such that:

(1) $a_1 = 1$;

(2):$|a_i - a_{i+1}| \leq 2$, $i = 1, \ldots, n-1$.

Determine whether $f(1996)$ is divisible by 3.

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4 Let triangle ABC be an isosceles triangle with $AB = AC$. Suppose that the angle bisector of its angle $\angle B$ meets the side AC at a point D and that $BC = BD + AD$.
Determine $\angle A$.

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5 Let $r_1, r_2, \ldots, r_m$ be a given set of m positive rational numbers such that $\sum_{k=1}^{m} r_k = 1$. Define the function f by $f(n) = n - \sum_{k=1}^{m} [r_k n]$ for each positive integer n. Determine the minimum and maximum values of $f(n)$. Here $[x]$ denotes the greatest integer less than or equal to x.
