Canada National Olympiad 1998

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1 Determine the number of real solutions a to the equation:

$$\left[\frac{1}{2}a\right]+\left[\frac{1}{3}a\right]+\left[\frac{1}{5}a\right]=a.$$

Here, if x is a real number, then $[\,x\,]$ denotes the greatest integer that is less than or equal to x.

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2 Find all real numbers x such that:

$$x=\sqrt{x-\frac{1}{x}}+\sqrt{1-\frac{1}{x}}$$

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3 Let n be a natural number such that $n\geq 2$. Show that

$$\frac{1}{n+1}\left(1+\frac{1}{3}+\cdots+\frac{1}{2n-1}\right)>\frac{1}{n}\left(\frac{1}{2}+\frac{1}{4}+\cdots+\frac{1}{2n}\right).$$

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4 Let ABC be a triangle with $\angle BAC=40^\circ$ and $\angle ABC=60^\circ$. Let D and E be the points lying on the sides AC and AB, respectively, such that $\angle CBD=40^\circ$ and $\angle BCE=70^\circ$. Let F be the point of intersection of the lines BD and CE. Show that the line AF is perpendicular to the line BC.

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5 Let m be a positive integer. Define the sequence $a_0,a_1,a_2,\cdots$ by $a_0=0,\ a_1=m$, and $a_{n+1}=m^2a_n-a_{n-1}$ for $n=1,2,3,\cdots$.
Prove that an ordered pair (a,b) of non-negative integers, with $a\leq b$, gives a solution to the equation

$$\frac{a^2+b^2}{ab+1}=m^2$$

if and only if (a,b) is of the form (a_n,a_{n+1}) for some $n\geq 0$.
