Canada National Olympiad 2000

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1 At 12:00 noon, Anne, Beth and Carmen begin running laps around a circular track of length 300 meters, all starting from the same point on the track. Each jogger maintains a constant speed in one of the two possible directions for an indefinite period of time. Show that if Anne's speed is different from the other two speeds, then at some later time Anne will be at least 100 meters from each of the other runners. (Here, distance is measured along the shorter of the two arcs separating two runners.)

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2 A *permutation* of the integers $1901, 1902, \cdots, 2000$ is a sequence $a_1, a_2, \cdots, a_{100}$ in which each of those integers appears exactly once. Given such a permutation, we form the sequence of partial sums

$$s_1 = a_1, \quad s_2 = a_1 + a_2, \quad s_3 = a_1 + a_2 + a_3, \quad \dots \quad, s_{100} = a_1 + a_2 + \cdots + a_{100}.$$

How many of these permutations will have no terms of the sequence $s_1, \dots, s_{100}$ divisible by three?

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3 Let $A = (a_1, a_2, \cdots, a_{2000})$ be a sequence of integers each lying in the interval $[-1000, 1000]$. Suppose that the entries in A sum to 1. Show that some nonempty subsequence of A sums to zero.

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4 Let $ABCD$ be a convex quadrilateral with $\angle CBD = 2\angle ADB$, $\angle ABD = 2\angle CDB$ and $AB = CB$.

Prove that $AD = CD$.

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5 Suppose that the real numbers $a_1, a_2, \dots, a_{100}$ satisfy

$$\begin{aligned} 0 \le a_{100} \le a_{99} \le \cdots \le a_2 &\le a_1, \\ a_1 + a_2 &\le 100 \\ a_3 + a_4 + \cdots + a_{100} &\le 100. \end{aligned}$$

Determine the maximum possible value of $a_1^2 + a_2^2 + \cdots + a_{100}^2$, and find all possible sequences $a_1, a_2, \dots, a_{100}$ which achieve this maximum.
