Cono Sur Olympiad 2016

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–	Day 1
1	Let $\overline{abcd}$ be one of the 9999 numbers $0001, 0002, 0003, \ldots, 9998, 9999$. Let $\overline{abcd}$ be an *special* number if $ab - cd$ and $ab + cd$ are perfect squares, $ab - cd$ divides $ab + cd$ and also $ab + cd$ divides $abcd$. For example 2016 is special. Find all the $\overline{abcd}$ special numbers. **Note:** If $\overline{abcd} = 0206$, then $ab = 02$ and $cd = 06$.
2	For every $k = 1, 2, \ldots$ let s_k be the number of pairs (x, y) satisfying the equation $kx + (k+1)y = 1001 - k$ with x, y non-negative integers. Find $s_1 + s_2 + \cdots + s_{200}$.
3	There are 2016 positions marked around a circle, with a token on one of them. A legitimate move is to move the token either 1 position or 4 positions from its location, clockwise. The restriction is that the token can not occupy the same position more than once. Players A and B take turns making moves. Player A has the first move. The first player who cannot make a legitimate move loses. Determine which of the two players has a winning strategy.
–	Day 2
4	Let $S(n)$ be the sum of the digits of the positive integer n. Find all n such that $S(n)(S(n)-1) = n - 1$.
5	Let ABC be a triangle inscribed on a circle with center O. Let D and E be points on the sides AB and BC,respectively, such that $AD = DE = EC$. Let X be the intersection of the angle bisectors of $\angle ADE$ and $\angle DEC$. If $X \neq O$, show that, the lines OX and DE are perpendicular.
6	We say that three different integers are *friendly* if one of them divides the product of the other two. Let n be a positive integer. a) Show that, between n^2 and $n^2 + n$, exclusive, does not exist any triplet of friendly numbers. b) Determine if for each n exists a triplet of friendly numbers between n^2 and $n^2 + n + 3\sqrt{n}$, exclusive.