

Canada National Olympiad 2009

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by BigSams

- 1 Given an $m \times n$ grid with unit squares coloured either black or white, a black square in the grid is *stranded* if there is some square to its left in the same row that is white and there is some square above it in the same column that is white.
Find a closed formula for the number of $2 \times n$ grids with no stranded black square.
Note that n is any natural number and the formula must be in terms of n with no other variables.

- 2 Two circles of different radii are cut out of cardboard. Each circle is subdivided into 200 equal sectors. On each circle 100 sectors are painted white and the other 100 are painted black. The smaller circle is then placed on top of the larger circle, so that their centers coincide. Show that one can rotate the small circle so that the sectors on the two circles line up and at least 100 sectors on the small circle lie over sectors of the same color on the big circle.

- 3 Define $f(x, y, z) = \frac{(xy+yz+zx)(x+y+z)}{(x+y)(y+z)(z+x)}$.
Determine the set of real numbers r for which there exists a triplet of positive real numbers satisfying $f(x, y, z) = r$.

- 4 Find all ordered pairs of integers (a, b) such that $3^a + 7^b$ is a perfect square.

- 5 A set of points is marked on the plane, with the property that any three marked points can be covered with a disk of radius 1. Prove that the set of all marked points can be covered with a disk of radius 1.