Canada National Olympiad 2012

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by WakeUp, mlm95

1   Let x, y and z be positive real numbers. Show that $x^2 + xy^2 + xyz^2 \geq 4xyz - 4$.

2   For any positive integers n and k, let $L(n, k)$ be the least common multiple of the k consecutive integers $n, n + 1, \ldots, n + k - 1$. Show that for any integer b, there exist integers n and k such that $L(n, k) > bL(n + 1, k)$.

3   Let $ABCD$ be a convex quadrilateral and let P be the point of intersection of AC and BD. Suppose that $AC + AD = BC + BD$. Prove that the internal angle bisectors of $\angle ACB$, $\angle ADB$ and $\angle APB$ meet at a common point.

4   A number of robots are placed on the squares of a finite, rectangular grid of squares. A square can hold any number of robots. Every edge of each square of the grid is classified as either passable or impassable. All edges on the boundary of the grid are impassable. You can give any of the commands up, down, left, or right.

      All of the robots then simultaneously try to move in the specified direction. If the edge adjacent to a robot in that direction is passable, the robot moves across the edge and into the next square. Otherwise, the robot remains on its current square. You can then give another command of up, down, left, or right, then another, for as long as you want. Suppose that for any individual robot, and any square on the grid, there is a finite sequence of commands that will move that robot to that square. Prove that you can also give a finite sequence of commands such that all of the robots end up on the same square at the same time.

5   A bookshelf contains n volumes, labelled 1 to n, in some order. The librarian wishes to put them in the correct order as follows. The librarian selects a volume that is too far to the right, say the volume with label k, takes it out, and inserts it in the k-th position. For example, if the bookshelf contains the volumes $1, 3, 2, 4$ in that order, the librarian could take out volume 2 and place it in the second position. The books will then be in the correct order $1, 2, 3, 4$.

      (a) Show that if this process is repeated, then, however the librarian makes the selections, all the volumes will eventually be in the correct order.
      (b) What is the largest number of steps that this process can take?