Canada National Olympiad 2013

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1 Determine all polynomials $P(x)$ with real coefficients such that

$$(x+1)P(x-1)-(x-1)P(x)$$

is a constant polynomial.

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2 The sequence $a_1, a_2, \dots, a_n$ consists of the numbers $1, 2, \dots, n$ in some order. For which positive integers n is it possible that the $n+1$ numbers $0, a_1, a_1+a_2, a_1+a_2+a_3, \dots, a_1+a_2+\cdots+a_n$ all have di fferent remainders when divided by $n+1$?

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3 Let G be the centroid of a right-angled triangle ABC with $\angle BCA = 90^\circ$. Let P be the point on ray AG such that $\angle CPA = \angle CAB$, and let Q be the point on ray BG such that $\angle CQB = \angle ABC$. Prove that the circumcircles of triangles AQG and BPG meet at a point on side AB.

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4 Let n be a positive integer. For any positive integer j and positive real number r, define $f_j(r)$ and $g_j(r)$ by

$$f_j(r) = \min(jr, n) + \min\left(\frac{j}{r}, n\right), \text{ and } g_j(r) = \min(\lceil jr \rceil, n) + \min\left(\left\lceil \frac{j}{r} \right\rceil, n\right),$$

where $\lceil x \rceil$ denotes the smallest integer greater than or equal to x. Prove that

$$\sum_{j=1}^{n} f_j(r) \le n^2 + n \le \sum_{j=1}^{n} g_j(r)$$

for all positive real numbers r.

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5 Let O denote the circumcentre of an acute-angled triangle ABC. Let point P on side AB be such that $\angle BOP = \angle ABC$, and let point Q on side AC be such that $\angle COQ = \angle ACB$. Prove that the reflection of BC in the line PQ is tangent to the circumcircle of triangle APQ.
