Germany Team Selection Test 1978

www.artofproblemsolving.com/community/c5065

by Amir Hossein

1 Let E be a set of n points in the plane $(n \geq 3)$ whose coordinates are integers such that any three points from E are vertices of a nondegenerate triangle whose centroid doesnt have both coordinates integers. Determine the maximal $n.$

---

2 Let S be a convex quadrilateral $ABCD$ and O a point inside it. The feet of the perpendiculars from O to AB, BC, CD, DA are A_1, B_1, C_1, D_1 respectively. The feet of the perpendiculars from O to the sides of S_i, the quadrilateral $A_iB_iC_iD_i$, are $A_{i+1}B_{i+1}C_{i+1}D_{i+1}$, where $i = 1, 2, 3.$ Prove that S_4 is similar to S.

---

3 Let n be an integer greater than 1. Define

$$x_1 = n, y_1 = 1, x_{i+1} = \left[\frac{x_i + y_i}{2}\right], y_{i+1} = \left[\frac{n}{x_{i+1}}\right], \qquad \text{for } i = 1, 2, \ldots\ ,$$

where $[z]$ denotes the largest integer less than or equal to z. Prove that

$$\min\{x_1, x_2, \ldots, x_n\} = [\sqrt{n}]$$

---

4 Let B be a set of k sequences each having n terms equal to 1 or -1. The product of two such sequences $(a_1, a_2, \ldots, a_n)$ and $(b_1, b_2, \ldots, b_n)$ is defined as $(a_1b_1, a_2b_2, \ldots, a_nb_n)$. Prove that there exists a sequence $(c_1, c_2, \ldots, c_n)$ such that the intersection of B and the set containing all sequences from B multiplied by $(c_1, c_2, \ldots, c_n)$ contains at most $\frac{k^2}{2^n}$ sequences.

---

5 Let E be a finite set of points such that E is not contained in a plane and no three points of E are collinear. Show that at least one of the following alternatives holds:

(i) E contains five points that are vertices of a convex pyramid having no other points in common with $E;$

(ii) some plane contains exactly three points from $E.$

---

6 A lattice point in the plane is a point both of whose coordinates are integers. Each lattice point has four neighboring points: upper, lower, left, and right. Let k be a circle with radius $r \geq 2$, that does not pass through any lattice point. An interior boundary point is a lattice point lying inside the circle k that has a neighboring point lying outside k. Similarly, an exterior boundary point is a lattice point lying outside the circle k that has a neighboring point lying inside k. Prove that there are four more exterior boundary points than interior boundary points.

