Germany Team Selection Test 2002

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–	VAIMO 1
1	Determine the number of all numbers which are represented as x^2+y^2 with $x, y \in \{1, 2, 3, \ldots, 1000\}$ and which are divisible by 121.
2	Prove: If x, y, z are the lengths of the angle bisectors of a triangle with perimeter 6, than we have: $$\frac{1}{x^2}+\frac{1}{y^2}+\frac{1}{z^2} \geq 1.$$
3	Determine all $(x,y) \in \mathbb{N}^2$ which satisfy $x^{2y}+(x+1)^{2y}=(x+2)^{2y}$.
–	VAIMO 2
1	Let P denote the set of all ordered pairs (p,q) of nonnegative integers. Find all functions $f: P \to \mathbb{R}$ satisfying $$f(p,q)=\begin{cases} 0 & \text{if } pq=0, \\ 1+\frac{1}{2}f(p+1,q-1)+\frac{1}{2}f(p-1,q+1) & \text{otherwise} \end{cases}$$ Compare IMO shortlist problem 2001, algebra A1 for the three-variable case.
2	Let A_1 be the center of the square inscribed in acute triangle ABC with two vertices of the square on side BC. Thus one of the two remaining vertices of the square is on side AB and the other is on AC. Points $B_1,\ C_1$ are defined in a similar way for inscribed squares with two vertices on sides AC and AB, respectively. Prove that lines $AA_1,\ BB_1,\ CC_1$ are concurrent.
3	Prove that there is no positive integer n such that, for $k=1,2,\ldots,9$, the leftmost digit (in decimal notation) of $(n+k)!$ equals k.