Germany Team Selection Test 2003

www.artofproblemsolving.com/community/c5067

by orl, Philip_Leszczynski, grobber

– VAIMO 1

1 At a chess tournament the winner gets 1 point and the defeated one 0 points. A tie makes both obtaining $\frac{1}{2}$ points. 14 players, none of them equally aged, participated in a competition where everybody played against all the other players. After the competition a ranking was carried out. Of the two players with the same number of points the younger received the better ranking. After the competition Jan realizes that the best three players together got as many points as the last 9 players obtained points together. And Joerg noted that the number of ties was maximal. Determine the number of ties.

2 Given a triangle ABC and a point M such that the lines MA, MB, MC intersect the lines BC, CA, AB in this order in points D, E and F, respectively. Prove that there are numbers $\epsilon_1, \epsilon_2, \epsilon_3 \in \{-1, 1\}$ such that:

$$\epsilon_1 \cdot \frac{MD}{AD} + \epsilon_2 \cdot \frac{ME}{BE} + \epsilon_3 \cdot \frac{MF}{CF} = 1.$$

3 Let N be a natural number and $x_1, \ldots, x_n$ further natural numbers less than N and such that the least common multiple of any two of these n numbers is greater than N. Prove that the sum of the reciprocals of these n numbers is always less than 2: $\sum_{i=1}^n \frac{1}{x_i} < 2$.

– VAIMO 2

1 Find all functions f from the reals to the reals such that

$$f\left(f(x) + y\right) = 2x + f\left(f(y) - x\right)$$

for all real x, y.

2 Let B be a point on a circle S_1, and let A be a point distinct from B on the tangent at B to S_1. Let C be a point not on S_1 such that the line segment AC meets S_1 at two distinct points. Let S_2 be the circle touching AC at C and touching S_1 at a point D on the opposite side of AC from B. Prove that the circumcentre of triangle BCD lies on the circumcircle of triangle ABC.

3 For n an odd positive integer, the unit squares of an $n \times n$ chessboard are coloured alternately black and white, with the four corners coloured black. A it tromino is an L-shape formed by three connected unit squares. For which values of n is it possible to cover all the black squares with non-overlapping trominos? When it is possible, what is the minimum number of trominos needed?