Germany Team Selection Test 2005

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Day 1

1 Given the positive numbers a and b and the natural number n, find the greatest among the $n+1$ monomials in the binomial expansion of $(a+b)^n$.

2 Let M be a set of points in the Cartesian plane, and let (S) be a set of segments (whose endpoints not necessarily have to belong to M) such that one can walk from any point of M to any other point of M by travelling along segments which are in (S). Find the smallest total length of the segments of (S) in the cases

a.) $M=\{(-1,0),(0,0),(1,0),(0,-1),(0,1)\}$.
b.) $M=\{(-1,-1),(-1,0),(-1,1),(0,-1),(0,0),(0,1),(1,-1),(1,0),(1,1)\}$.

In other words, find the Steiner trees of the set M in the above two cases.

3 Let a, b, c, d and n be positive integers such that $7\cdot 4^n = a^2+b^2+c^2+d^2$. Prove that the numbers a, b, c, d are all $\geq 2^{n-1}$.

Day 2

1 Let a_0, a_1, a_2, ... be an infinite sequence of real numbers satisfying the equation $a_n = |a_{n+1} - a_{n+2}|$ for all $n \geq 0$, where a_0 and a_1 are two different positive reals.

Can this sequence a_0, a_1, a_2, ... be bounded?

Proposed by Mihai Blun, Romania

2 Let Γ be a circle and let d be a line such that Γ and d have no common points. Further, let AB be a diameter of the circle Γ; assume that this diameter AB is perpendicular to the line d, and the point B is nearer to the line d than the point A. Let C be an arbitrary point on the circle Γ, different from the points A and B. Let D be the point of intersection of the lines AC and d. One of the two tangents from the point D to the circle Γ touches this circle Γ at a point E; hereby, we assume that the points B and E lie in the same halfplane with respect to the line AC. Denote by F the point of intersection of the lines BE and d. Let the line AF intersect the circle Γ at a point G, different from A.

Prove that the reflection of the point G in the line AB lies on the line CF.

3 Let n and k be positive integers. There are given n circles in the plane. Every two of them intersect at two distinct points, and all points of intersection they determine are pairwise distinct (i. e. no three circles have a common point). No three circles have a point in common. Each intersection point must be colored with one of n distinct colors so that each color is used at least once and exactly k distinct colors occur on each circle. Find all values of $n \geq 2$ and k for which such a coloring is possible.

Proposed by Horst Sewerin, Germany

Day 3 February 12th

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1 Prove that there doesn't exist any positive integer n such that $2n^2+1, 3n^2+1$ and $6n^2+1$ are perfect squares.

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2 If a, b, c are positive reals such that $a+b+c=1$, prove that

$$\frac{1+a}{1-a}+\frac{1+b}{1-b}+\frac{1+c}{1-c} \leq 2\left(\frac{b}{a}+\frac{c}{b}+\frac{a}{c}\right).$$

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3 Let ABC be a triangle and let r, r_a, r_b, r_c denote the inradius and ex-radii opposite to the vertices A, B, C, respectively. Suppose that $a > r_a, b > r_b, c > r_c$. Prove that
(a) $\triangle ABC$ is acute.
(b) $a+b+c > r+r_a+r_b+r_c$.

Day 4 February 15th

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1 Let $\tau(n)$ denote the number of positive divisors of the positive integer n. Prove that there exist infinitely many positive integers a such that the equation $\tau(an)=n$ does not have a positive integer solution n.

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2 If a, b ,c are three positive real numbers such that $ab+bc+ca=1$, prove that

$$\sqrt[3]{\frac{1}{a}+6b}+\sqrt[3]{\frac{1}{b}+6c}+\sqrt[3]{\frac{1}{c}+6a} \leq \frac{1}{abc}.$$

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3 For an $n\times n$ matrix A, let X_i be the set of entries in row i, and Y_j the set of entries in column j, $1\leq i,j\leq n$. We say that A is *golden* if $X_1,\dots,X_n,Y_1,\dots,Y_n$ are distinct sets. Find the least integer n such that there exists a 2004×2004 golden matrix with entries in the set $\{1,2,\dots,n\}$.

Day 5 March 20th

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1 Let k be a fixed integer greater than 1, and let $m = 4k^2 - 5$. Show that there exist positive integers a and b such that the sequence (x_n) defined by

$$x_0 = a, \quad x_1 = b, \quad x_{n+2} = x_{n+1} + x_n \quad \text{for} \quad n = 0, 1, 2, \dots,$$

has all of its terms relatively prime to m.

Proposed by Jaroslaw Wroblewski, Poland

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2 Let O be the circumcenter of an acute-angled triangle ABC with $\angle B < \angle C$. The line AO meets the side BC at D. The circumcenters of the triangles ABD and ACD are E and F, respectively. Extend the sides BA and CA beyond A, and choose on the respective extensions points G and H such that $AG = AC$ and $AH = AB$. Prove that the quadrilateral $EFGH$ is a rectangle if and only if $\angle ACB - \angle ABC = 60^\circ$.

Proposed by Hojoo Lee, Korea

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3 A positive integer is called *nice* if the sum of its digits in the number system with base 3 is divisible by 3.

Calculate the sum of the first 2005 nice positive integers.

Day 6 April 24th

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1 Find all monotonically increasing or monotonically decreasing functions $f : \mathbb{R}_+ \to \mathbb{R}_+$ which satisfy the equation $f(xy) \cdot f\left(\frac{f(y)}{x}\right) = 1$ for any two numbers x and y from $\mathbb{R}_+$.

Hereby, $\mathbb{R}_+$ is the set of all positive real numbers.

Note. A function $f : \mathbb{R}_+ \to \mathbb{R}_+$ is called *monotonically increasing* if for any two positive numbers x and y such that $x \geq y$, we have $f(x) \geq f(y)$.
A function $f : \mathbb{R}_+ \to \mathbb{R}_+$ is called *monotonically decreasing* if for any two positive numbers x and y such that $x \geq y$, we have $f(x) \leq f(y)$.

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2 Let ABC be a triangle satisfying $BC < CA$. Let P be an arbitrary point on the side AB (different from A and B), and let the line CP meet the circumcircle of triangle ABC at a point S (apart from the point C).

Let the circumcircle of triangle ASP meet the line CA at a point R (apart from A), and let the circumcircle of triangle BPS meet the line CB at a point Q (apart from B).

Prove that the excircle of triangle APR at the side AP is identical with the excircle of triangle PQB at the side PQ if and only if the point S is the midpoint of the arc AB on the circumcircle of triangle ABC.

3 Let b and c be any two positive integers. Define an integer sequence a_n, for $n \geq 1$, by $a_1 = 1$, $a_2 = 1$, $a_3 = b$ and $a_{n+3} = ba_{n+2}a_{n+1} + ca_n$.

Find all positive integers r for which there exists a positive integer n such that the number a_n is divisible by r.

Day 7 May 7th

1 In the following, a *word* will mean a finite sequence of letters "a" and "b". The *length* of a word will mean the number of the letters of the word. For instance, $abaab$ is a word of length 5. There exists exactly one word of length 0, namely the empty word.

A word w of length ℓ consisting of the letters x_1, x_2, ..., x_ℓ in this order is called a *palindrome* if and only if $x_j = x_{\ell+1-j}$ holds for every j such that $1 \leq j \leq \ell$. For instance, $baaab$ is a palindrome; so is the empty word.

For two words w_1 and w_2, let w_1w_2 denote the word formed by writing the word w_2 directly after the word w_1. For instance, if $w_1 = baa$ and $w_2 = bb$, then $w_1w_2 = baabb$.

Let r, s, t be nonnegative integers satisfying $r + s = t + 2$. Prove that there exist palindromes A, B, C with lengths r, s, t, respectively, such that $AB = Cab$, if and only if the integers $r + 2$ and $s - 2$ are coprime.

2 Let n be a positive integer, and let a_1, a_2, ..., a_n, b_1, b_2, ..., b_n be positive real numbers such that $a_1 \geq a_2 \geq ... \geq a_n$ and $b_1 \geq a_1$, $b_1b_2 \geq a_1a_2$, $b_1b_2b_3 \geq a_1a_2a_3$, ..., $b_1b_2...b_n \geq a_1a_2...a_n$.

Prove that $b_1 + b_2 + ... + b_n \geq a_1 + a_2 + ... + a_n$.

3 Let ABC be a triangle with area S, and let P be a point in the plane. Prove that $AP+BP+CP \geq 2\sqrt[4]{3}\sqrt{S}$.

Day 8 May 29th

1 **(a)** Does there exist a positive integer n such that the decimal representation of $n!$ ends with the string 2004, followed by a number of digits from the set $\{0;\ 4\}$?

(b) Does there exist a positive integer n such that the decimal representation of $n!$ starts with the string 2004 ?

2 For any positive integer n, prove that there exists a polynomial P of degree n such that all coeffients of this polynomial P are integers, and such that the numbers $P\left(0\right)$, $P\left(1\right)$, $P\left(2\right)$, ..., $P\left(n\right)$ are pairwisely distinct powers of 2.

3 Let ABC be a triangle with orthocenter H, incenter I and centroid S, and let d be the diameter of the circumcircle of triangle ABC. Prove the inequality

$$9 \cdot HS^2 + 4\left(AH \cdot AI + BH \cdot BI + CH \cdot CI\right) \geq 3d^2,$$

and determine when equality holds.

Day 9 May 30th

1 Find the smallest positive integer n with the following property:

For any integer m with $0 < m < 2004$, there exists an integer k such that

$$\frac{m}{2004} < \frac{k}{n} < \frac{m+1}{2005}.$$

2 Let n be a positive integer such that $n \geq 3$. Let a_1, a_2, ..., a_n and b_1, b_2, ..., b_n be $2n$ positive real numbers satisfying the equations

$$a_1 + a_2 + ... + a_n = 1, \quad \text{and} \quad b_1^2 + b_2^2 + ... + b_n^2 = 1.$$

Prove the inequality

$$a_1\left(b_1 + a_2\right) + a_2\left(b_2 + a_3\right) + ... + a_{n-1}\left(b_{n-1} + a_n\right) + a_n\left(b_n + a_1\right) < 1.$$

3 Prove that, among 32 integer numbers, one can always find 16 whose sum is divisible by 16.