Germany Team Selection Test 2006

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by Pentakratie, Nima Ahmadi Pour, Megus, Michal Marcinkowski, who, Amir.S, ZetaX, rohitsingh0812, epitomy01, andre.l, madatmath, darij grinberg, Maverick, jhaussmann5, carlosbr

Day 1

1 Let A, B, C, D, E, F be six points on a circle such that $AE \parallel BD$ and $BC \parallel DF$. Let X be the reflection of the point D in the line CE. Prove that the distance from the point X to the line EF equals to the distance from the point B to the line AC.

2 In a room, there are 2005 boxes, each of them containing one or several sorts of fruits, and of course an integer amount of each fruit.

a) Show that we can find 669 boxes, which altogether contain at least a third of all apples and at least a third of all bananas.

b) Can we always find 669 boxes, which altogether contain at least a third of all apples, at least a third of all bananas and at least a third of all pears?

3 Is the following statement true?

For each positive integer n, we can find eight nonnegative integers a, b, c, d, e, f, g, h such that $n = \frac{2^a-2^b}{2^c-2^d} \cdot \frac{2^e-2^f}{2^g-2^h}$.

Day 2

1 We denote by $\mathbb{R}^+$ the set of all positive real numbers.

Find all functions $f: \mathbb{R}^+ \rightarrow \mathbb{R}^+$ which have the property:

$$f(x)f(y) = 2f(x + yf(x))$$

for all positive real numbers x and y.

Proposed by Nikolai Nikolov, Bulgaria

2 Given a triangle ABC satisfying $AC + BC = 3 \cdot AB$. The incircle of triangle ABC has center I and touches the sides BC and CA at the points D and E, respectively. Let K and L be the reflections of the points D and E with respect to I. Prove that the points A, B, K, L lie on one circle.

Proposed by Dimitris Kontogiannis, Greece

3 Consider a $m \times n$ rectangular board consisting of mn unit squares. Two of its unit squares are called *adjacent* if they have a common edge, and a *path* is a sequence of unit squares in which any two consecutive squares are adjacent. Two parths are called *non-intersecting* if they don't share any common squares.

Each unit square of the rectangular board can be colored black or white. We speak of a *coloring* of the board if all its mn unit squares are colored.

Let N be the number of colorings of the board such that there exists at least one black path from the left edge of the board to its right edge. Let M be the number of colorings of the board for which there exist at least two non-intersecting black paths from the left edge of the board to its right edge.

Prove that $N^2 \geq M \cdot 2^{mn}$.

Day 3

1 A house has an even number of lamps distributed among its rooms in such a way that there are at least three lamps in every room. Each lamp shares a switch with exactly one other lamp, not necessarily from the same room. Each change in the switch shared by two lamps changes their states simultaneously. Prove that for every initial state of the lamps there exists a sequence of changes in some of the switches at the end of which each room contains lamps which are on as well as lamps which are off.

Proposed by Australia

2 In an acute triangle ABC, let D, E, F be the feet of the perpendiculars from the points A, B, C to the lines BC, CA, AB, respectively, and let P, Q, R be the feet of the perpendiculars from the points A, B, C to the lines EF, FD, DE, respectively.

Prove that $p\left(ABC\right)p\left(PQR\right) \geq \left(p\left(DEF\right)\right)^{2}$, where $p\left(T\right)$ denotes the perimeter of triangle T .

Proposed by Hojoo Lee, Korea

Day 4

1 Let a, b, c, d, e, f be positive integers and let $S = a+b+c+d+e+f$.
Suppose that the number S divides $abc+def$ and $ab+bc+ca-de-ef-df$. Prove that S is composite.

2 Four real numbers p, q, r, s satisfy $p+q+r+s = 9$ and $p^2+q^2+r^2+s^2 = 21$. Prove that there exists a permutation (a,b,c,d) of (p,q,r,s) such that $ab-cd \geq 2$.

3 Suppose we have a n-gon. Some $n-3$ diagonals are coloured black and some other $n-3$ diagonals are coloured red (a side is not a diagonal), so that no two diagonals of the same colour can intersect strictly inside the polygon, although they can share a vertex. Find the maximum number of intersection points between diagonals coloured differently strictly inside the polygon, in terms of n.

Proposed by Alexander Ivanov, Bulgaria

Day 5

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1 Does there exist a natural number n in whose decimal representation each digit occurs at least 2006 times and which has the property that you can find two different digits in its decimal representation such that the number obtained from n by interchanging these two digits is different from n and has the same set of prime divisors as n ?

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2 Find all functions $f:\mathbb{R}\to\mathbb{R}$ such that $f(x+y)+f(x)f(y)=f(xy)+2xy+1$ for all real numbers x and y.

Proposed by B.J. Venkatachala, India

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3 Let $ABCD$ be a parallelogram. A variable line g through the vertex A intersects the rays BC and DC at the points X and Y, respectively. Let K and L be the A-excenters of the triangles ABX and ADY. Show that the angle $\measuredangle KCL$ is independent of the line g.

Proposed by Vyacheslev Yasinskiy, Ukraine

Day 6

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1 For any positive integer n, let $w\left(n\right)$ denote the number of different prime divisors of the number n. (For instance, $w\left(12\right)=2$.) Show that there exist infinitely many positive integers n such that $w\left(n\right)<w\left(n+1\right)<w\left(n+2\right)$.

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2 Let A_1, B_1, C_1 be the feet of the altitudes of an acute-angled triangle ABC issuing from the vertices A, B, C, respectively. Let K and M be points on the segments A_1C_1 and B_1C_1, respectively, such that $\measuredangle KAM=\measuredangle A_1AC$. Prove that the line AK is the angle bisector of the angle C_1KM.

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3 Does there exist a set M of points in space such that every plane intersects M at a finite but nonzero number of points?

Day 7

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1 Let $n\geq 3$ be a fixed integer. Each side and each diagonal of a regular n-gon is labelled with a number from the set $\left\{1;\;2;\;...;\;r\right\}$ in a way such that the following two conditions are fulfilled:

1. Each number from the set $\{1;\ 2;\ ...;\ r\}$ occurs at least once as a label.

2. In each triangle formed by three vertices of the n-gon, two of the sides are labelled with the same number, and this number is greater than the label of the third side.

(a) Find the maximal r for which such a labelling is possible.

(b) *Harder version (IMO Shortlist 2005):* For this maximal value of r, how many such labellings are there?

Easier version (5th German TST 2006): Show that, for this maximal value of r, there are exactly $\frac{n!(n-1)!}{2^{n-1}}$ possible labellings.

Proposed by Federico Ardila, Colombia

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2 Find all positive integers n such that there exists a unique integer a such that $0 \leq a < n!$ with the following property:

$$n! \mid a^n + 1$$

Proposed by Carlos Caicedo, Colombia

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3 The diagonals AC and BD of a cyclic quadrilateral $ABCD$ meet at a point X. The circumcircles of triangles ABX and CDX meet at a point Y (apart from X). Let O be the center of the circumcircle of the quadrilateral $ABCD$. Assume that the points O, X, Y are all distinct. Show that OY is perpendicular to XY.

Day 8

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1 Let ABC be an equilateral triangle, and P, Q, R three points in its interior satisfying

$$\measuredangle PCA = \measuredangle CAR = 15^\circ,\ \measuredangle RBC = \measuredangle BCQ = 20^\circ,\ \measuredangle QAB = \measuredangle ABP = 25^\circ.$$

Compute the angles of triangle PQR.

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2 There are n markers, each with one side white and the other side black. In the beginning, these n markers are aligned in a row so that their white sides are all up. In each step, if possible, we choose a marker whose white side is up (but not one of the outermost markers), remove it, and reverse the closest marker to the left of it and also reverse the closest marker to the right of it. Prove that, by a finite sequence of such steps, one can achieve a state with only two markers remaining if and only if $n-1$ is not divisible by 3.

Proposed by Dusan Dukic, Serbia

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3 Let n be a positive integer, and let b_1, b_2, ..., b_n be n positive reals. Set $a_1 = \frac{b_1}{b_1+b_2+...+b_n}$ and $a_k = \frac{b_1+b_2+...+b_k}{b_1+b_2+...+b_{k-1}}$ for every $k > 1$. Prove the inequality
$a_1 + a_2 + ... + a_n \leq \frac{1}{a_1} + \frac{1}{a_2} + ... + \frac{1}{a_n}$.

Day 9

1 Find all real solutions x of the equation
$\cos\cos\cos\cos x = \sin\sin\sin\sin x$.

(Angles are measured in radians.)

2 The lengths of the altitudes of a triangle are positive integers, and the length of the radius of the incircle is a prime number.
Find the lengths of the sides of the triangle.

3 Suppose that a_1, a_2, . . ., a_n are integers such that $n \mid a_1 + a_2 + \ldots + a_n$.
Prove that there exist two permutations $(b_1, b_2, \ldots, b_n)$ and $(c_1, c_2, \ldots, c_n)$ of $(1, 2, \ldots, n)$ such that for each integer i with $1 \leq i \leq n$, we have

$$n \mid a_i - b_i - c_i$$

Proposed by Ricky Liu & Zuming Feng, USA