**Germany Team Selection Test 2008**

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by orl, Dida Drogbier, April, freemind, delegat, Omid Hatami

– VAIMO

**Day 1**

**1** Show that there is a digit unequal to 2 in the decimal represesentation of $\sqrt[3]{3}$ between the $1000000$-th und $3141592$-th position after decimal point.

**2** Let $ABCD$ be an isosceles trapezium with $AB||CD$ and $\bar{BC} = \bar{AD}$. The parallel to $AD$ through $B$ meets the perpendicular to $AD$ through $D$ in point $X$. The line through $A$ drawn which is parallel to $BD$ meets the perpendicular to $BD$ through $D$ in point $Y$. Prove that points $C, X, D$ and $Y$ lie on a common circle.

**3** Prove there is an integer $k$ for which $k^3 - 36k^2 + 51k - 97$ is a multiple of $3^{2008.}$

**Day 2**

**1** Find all pairs of natural numbers $(a, b)$ such that $7^a - 3^b$ divides $a^4 + b^2$.

*Author: Stephan Wagner, Austria*

**2** The diagonals of a trapezoid $ABCD$ intersect at point $P$. Point $Q$ lies between the parallel lines $BC$ and $AD$ such that $\angle AQD = \angle CQB$, and line $CD$ separates points $P$ and $Q$. Prove that $\angle BQP = \angle DAQ$.

*Author: Vyacheslav Yasinskiy, Ukraine*

**3** A rectangle $D$ is partitioned in several ($\geq 2$) rectangles with sides parallel to those of $D$. Given that any line parallel to one of the sides of $D$, and having common points with the interior of $D$, also has common interior points with the interior of at least one rectangle of the partition; prove that there is at least one rectangle of the partition having no common points with $D$'s boundary.

*Author: Kei Irie, Japan*

– AIMO 1

**1** Let $n > 1$ be an integer. Find all sequences $a_1, a_2, \ldots a_{n^2+n}$ satisfying the following conditions:

(a) $a_i \in \{0, 1\}$ for all $1 \leq i \leq n^2 + n$;

(b) $a_{i+1} + a_{i+2} + \ldots + a_{i+n} < a_{i+n+1} + a_{i+n+2} + \ldots + a_{i+2n}$ for all $0 \leq i \leq n^2 - n$.

*Author: Dusan Dukic, Serbia*

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**2** **(i)** Determine the smallest number of edges which a graph of $n$ nodes may have given that adding an arbitrary new edge would give rise to a 3-clique (3 nodes joined pairwise by edges).
**(ii)** Determine the smallest number of edges which a graph of $n$ nodes may have given that adding an arbitrary new edge would give rise to a 4-clique (4 nodes joined pairwise by edges).

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**3** Denote by $M$ midpoint of side $BC$ in an isosceles triangle $\triangle ABC$ with $AC = AB$. Take a point $X$ on a smaller arc $\overset{\frown}{MA}$ of circumcircle of triangle $\triangle ABM$. Denote by $T$ point inside of angle $BMA$ such that $\angle TMX = 90$ and $TX = BX$.

Prove that $\angle MTB - \angle CTM$ does not depend on choice of $X$.

*Author: Farzan Barekat, Canada*

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**–** AIMO 2

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**1** Consider those functions $f : \mathbb{N} \mapsto \mathbb{N}$ which satisfy the condition

$$f(m+n) \geq f(m) + f(f(n)) - 1$$

for all $m, n \in \mathbb{N}$. Find all possible values of $f(2007)$.

*Author: Nikolai Nikolov, Bulgaria*

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**2** Find all positive integers $n$ for which the numbers in the set $S = \{1, 2, \ldots, n\}$ can be colored red and blue, with the following condition being satisfied: The set $S \times S \times S$ contains exactly $2007$ ordered triples $(x, y, z)$ such that:

**(i)** the numbers $x$, $y$, $z$ are of the same color,
and
**(ii)** the number $x + y + z$ is divisible by $n$.

*Author: Gerhard Wginger, Netherlands*

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**3** Let $X$ be a set of 10,000 integers, none of them is divisible by 47. Prove that there exists a 2007-element subset $Y$ of $X$ such that $a - b + c - d + e$ is not divisible by 47 for any $a, b, c, d, e \in Y$.

*Author: Gerhard Wginger, Netherlands*

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**–** AIMO 3

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**1** Let $A_0 = (a_1, \ldots, a_n)$ be a finite sequence of real numbers. For each $k \geq 0$, from the sequence $A_k = (x_1, \ldots, x_k)$ we construct a new sequence $A_{k+1}$ in the following way.
1. We choose a partition $\{1, \ldots, n\} = I \cup J$, where $I$ and $J$ are two disjoint sets, such that the expression

$$\left| \sum_{i \in I} x_i - \sum_{j \in J} x_j \right|$$

attains the smallest value. (We allow $I$ or $J$ to be empty; in this case the corresponding sum is 0.) If there are several such partitions, one is chosen arbitrarily.
2. We set $A_{k+1} = (y_1, \ldots, y_n)$ where $y_i = x_i + 1$ if $i \in I$, and $y_i = x_i - 1$ if $i \in J$.
Prove that for some $k$, the sequence $A_k$ contains an element $x$ such that $|x| \geq \frac{n}{2}$.

*Author: Omid Hatami, Iran*

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**2** Let $ABC$ be a fixed triangle, and let $A_1$, $B_1$, $C_1$ be the midpoints of sides $BC$, $CA$, $AB$, respectively. Let $P$ be a variable point on the circumcircle. Let lines $PA_1$, $PB_1$, $PC_1$ meet the circumcircle again at $A'$, $B'$, $C'$, respectively. Assume that the points $A$, $B$, $C$, $A'$, $B'$, $C'$ are distinct, and lines $AA'$, $BB'$, $CC'$ form a triangle. Prove that the area of this triangle does not depend on $P$.

*Author: Christopher Bradley, United Kingdom*

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**3** Find all surjective functions $f : \mathbb{N} \to \mathbb{N}$ such that for every $m, n \in \mathbb{N}$ and every prime $p$, the number $f(m+n)$ is divisible by $p$ if and only if $f(m) + f(n)$ is divisible by $p$.

*Author: Mohsen Jamaali and Nima Ahmadi Pour Anari, Iran*

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**–** AIMO 4

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**1** Determine $Q \in \mathbb{R}$ which is so big that a sequence with non-negative reals elements $a_1, a_2, \ldots$ which satisfies the following two conditions:

**(i)** $\forall m, n \geq 1$ we have $a_{m+n} \leq 2\left(a_m + a_n\right)$
**(ii)** $\forall k \geq 0$ we have $a_{2^k} \leq \frac{1}{(k+1)^{2008}}$

such that for each sequence element we have the inequality $a_n \leq Q$.

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**2** Tracey baked a square cake whose surface is dissected in a $10 \times 10$ grid. In some of the fields she wants to put a strawberry such that for each four fields that compose a rectangle whose edges run in parallel to the edges of the cake boundary there is at least one strawberry. What is the minimum number of required strawberries?

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**3** Let $ABCD$ be an isosceles trapezium. Determine the geometric location of all points $P$ such that

$$|PA| \cdot |PC| = |PB| \cdot |PD|.$$

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**–** AIMO 5

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**1** A sequence $(S_n), n \geq 1$ of sets of natural numbers with $S_1 = \{1\}, S_2 = \{2\}$ and

$$S_{n+1} = \{k \in \mathbb{N} | k - 1 \in S_n \text{ XOR } k \in S_{n-1}\}.$$

Determine $S_{1024}$.

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**2** For three points $X, Y, Z$ let $R_{XYZ}$ be the circumcircle radius of the triangle $XYZ$. If $ABC$ is a triangle with incircle centre $I$ then we have:

$$\frac{1}{R_{ABI}} + \frac{1}{R_{BCI}} + \frac{1}{R_{CAI}} \leq \frac{1}{\bar{AI}} + \frac{1}{\bar{BI}} + \frac{1}{\bar{CI}}.$$

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**3** Find all real polynomials $f$ with $x, y \in \mathbb{R}$ such that

$$2yf(x+y) + (x-y)(f(x) + f(y)) \geq 0.$$

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**–** AIMO 6

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**1** Let $ABC$ be an acute triangle, and $M_a$, $M_b$, $M_c$ be the midpoints of the sides $a$, $b$, $c$. The perpendicular bisectors of $a$, $b$, $c$ (passing through $M_a$, $M_b$, $M_c$) intersect the boundary of the triangle again in points $T_a$, $T_b$, $T_c$. Show that if the set of points $\{A, B, C\}$ can be mapped to the set $\{T_a, T_b, T_c\}$ via a similitude transformation, then two feet of the altitudes of triangle $ABC$ divide the respective triangle sides in the same ratio. (Here, ”ratio” means the length of the shorter (or equal) part divided by the length of the longer (or equal) part.) Does the converse statement hold?

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**2** For every integer $k \geq 2,$ prove that $2^{3k}$ divides the number

$$\binom{2^{k+1}}{2^k} - \binom{2^k}{2^{k-1}}$$

but $2^{3k+1}$ does not.

*Author: Waldemar Pompe, Poland*

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**3** Determine all functions $f : \mathbb{R} \mapsto \mathbb{R}$ with $x, y \in \mathbb{R}$ such that

$$f(x - f(y)) = f(x + y) + f(y)$$

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**–** AIMO 7

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**1** Let $a_1, a_2, \ldots, a_{100}$ be nonnegative real numbers such that $a_1^2 + a_2^2 + \ldots + a_{100}^2 = 1$. Prove that

$$a_1^2 \cdot a_2 + a_2^2 \cdot a_3 + \ldots + a_{100}^2 \cdot a_1 < \frac{12}{25}.$$

*Author: Marcin Kuzma, Poland*

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**2** Point $P$ lies on side $AB$ of a convex quadrilateral $ABCD$. Let $\omega$ be the incircle of triangle $CPD$, and let $I$ be its incenter. Suppose that $\omega$ is tangent to the incircles of triangles $APD$ and $BPC$ at points $K$ and $L$, respectively. Let lines $AC$ and $BD$ meet at $E$, and let lines $AK$ and $BL$ meet at $F$. Prove that points $E$, $I$, and $F$ are collinear.

*Author: Waldemar Pompe, Poland*

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**3** Given is a convex polygon $P$ with $n$ vertices. Triangle whose vertices lie on vertices of $P$ is called *good* if all its sides are unit length. Prove that there are at most $\frac{2n}{3}$ *good* triangles.

*Author: Vyacheslav Yasinskiy, Ukraine*

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