Germany Team Selection Test 2009

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by April, orl

– TST 1

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1 In the plane we consider rectangles whose sides are parallel to the coordinate axes and have positive length. Such a rectangle will be called a *box*. Two boxes *intersect* if they have a common point in their interior or on their boundary. Find the largest n for which there exist n boxes B_1, $\ldots$, B_n such that B_i and B_j intersect if and only if $i \not\equiv j \pm 1 \pmod{n}$.

Proposed by Gerhard Woeginger, Netherlands

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2 Let k and n be integers with $0 \leq k \leq n - 2$. Consider a set L of n lines in the plane such that no two of them are parallel and no three have a common point. Denote by I the set of intersections of lines in L. Let O be a point in the plane not lying on any line of L. A point $X \in I$ is colored red if the open line segment OX intersects at most k lines in L. Prove that I contains at least $\frac{1}{2}(k + 1)(k + 2)$ red points.

Proposed by Gerhard Woeginger, Netherlands

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3 Let a, b, c, d be positive real numbers such that $abcd = 1$ and $a + b + c + d > \frac{a}{b} + \frac{b}{c} + \frac{c}{d} + \frac{d}{a}$. Prove that

$$a + b + c + d < \frac{b}{a} + \frac{c}{b} + \frac{d}{c} + \frac{a}{d}$$

Proposed by Pavel Novotn, Slovakia

– TST 2

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1 Given trapezoid $ABCD$ with parallel sides AB and CD, assume that there exist points E on line BC outside segment BC, and F inside segment AD such that $\angle DAE = \angle CBF$. Denote by I the point of intersection of CD and EF, and by J the point of intersection of AB and EF. Let K be the midpoint of segment EF, assume it does not lie on line AB. Prove that I belongs to the circumcircle of ABK if and only if K belongs to the circumcircle of CDJ.

Proposed by Charles Leytem, Luxembourg

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2 Let a_1, a_2, $\ldots$, a_n be distinct positive integers, $n \geq 3$. Prove that there exist distinct indices i and j such that $a_i + a_j$ does not divide any of the numbers $3a_1$, $3a_2$, $\ldots$, $3a_n$.

Proposed by Mohsen Jamaali, Iran

3 Let $S = \{x_1, x_2, \dots, x_{k+l}\}$ be a $(k+l)$-element set of real numbers contained in the interval $[0,1]$; k and l are positive integers. A k-element subset $A \subset S$ is called *nice* if

$$\left|\frac{1}{k}\sum_{x_i \in A} x_i - \frac{1}{l}\sum_{x_j \in S \setminus A} x_j\right| \le \frac{k+l}{2kl}$$

Prove that the number of nice subsets is at least $\dfrac{2}{k+l}\dbinom{k+l}{k}$.

Proposed by Andrey Badzyan, Russia

– TST 3

1 Let $ABCD$ be a chordal/cyclic quadrilateral. Consider points P, Q on AB and R, S on CD with

$$\overline{AP} : \overline{PB} = \overline{CS} : \overline{SD}, \quad \overline{AQ} : \overline{QB} = \overline{CR} : \overline{RD}.$$

How to choose P, Q, R, S such that $\overline{PR} \cdot \overline{AB} + \overline{QS} \cdot \overline{CD}$ is minimal?

2 Let $(a_n)_{n \in \mathbb{N}}$ defined by $a_1 = 1$, and $a_{n+1} = a_n^4 - a_n^3 + 2a_n^2 + 1$ for $n \ge 1$. Show that there is an infinite number of primes p such that none of the a_n is divisible by p.

3 Find all functions $f : \mathbb{R} \mapsto \mathbb{R}$ such that $\forall x, y, z \in \mathbb{R}$ we have: If

$$x^3 + f(y) \cdot x + f(z) = 0,$$

then

$$f(x)^3 + y \cdot f(x) + z = 0.$$

– TST 4

1 Let I be the incircle centre of triangle ABC and ω be a circle within the same triangle with centre I. The perpendicular rays from I on the sides $\overline{BC}, \overline{CA}$ and $\overline{AB}$ meets ω in A', B' and C'. Show that the three lines AA', BB' and CC' have a common point.

2 Tracy has been baking a rectangular cake whose surface is dissected by grid lines in square fields. The number of rows is 2^n and the number of columns is 2^{n+1} where $n \ge 1, n \in \mathbb{N}$. Now she covers the fields with strawberries such that each row has at least $2n+2$ of them. Show that there four pairwise distinct strawberries A, B, C and D which satisfy those three conditions:

(a) Strawberries A and B lie in the same row and A further left than B. Similarly D lies in the same row as C but further left.

(b) Strawberries B and C lie in the same column.

(c) Strawberries A lies further up and further left than D.

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3 Prove that for any four positive real numbers a, b, c, d the inequality

$$\frac{(a-b)(a-c)}{a+b+c}+\frac{(b-c)(b-d)}{b+c+d}+\frac{(c-d)(c-a)}{c+d+a}+\frac{(d-a)(d-b)}{d+a+b}\geq 0$$

holds. Determine all cases of equality.

Author: Darij Grinberg (Problem Proposal), Christian Reiher (Solution), Germany

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– TST 5

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1 Consider cubes of edge length 5 composed of 125 cubes of edge length 1 where each of the 125 cubes is either coloured black or white. A cube of edge length 5 is called "big", a cube od edge length is called "small". A posititve integer n is called "representable" if there is a big cube with exactly n small cubes where each row of five small cubes has an even number of black cubes whose centres lie on a line with distances $1, 2, 3, 4$ (zero counts as even number).

(a) What is the smallest and biggest representable number?
(b) Construct 45 representable numbers.

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2 Let $S \subseteq \mathbb{R}$ be a set of real numbers. We say that a pair (f,g) of functions from S into S is a *Spanish Couple* on S, if they satisfy the following conditions:
(i) Both functions are strictly increasing, i.e. $f(x) < f(y)$ and $g(x) < g(y)$ for all x, $y \in S$ with $x < y$;

(ii) The inequality $f\left(g\left(g\left(x\right)\right)\right) < g\left(f\left(x\right)\right)$ holds for all $x \in S$.

Decide whether there exists a Spanish Couple - on the set $S = \mathbb{N}$ of positive integers; - on the set $S = \{a - \frac{1}{b} : a, b \in \mathbb{N}\}$

Proposed by Hans Zantema, Netherlands

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3 In an acute triangle ABC segments BE and CF are altitudes. Two circles passing through the point A and F and tangent to the line BC at the points P and Q so that B lies between C and Q. Prove that lines PE and QF intersect on the circumcircle of triangle AEF.

Proposed by Davood Vakili, Iran

– TST 6

1 In the coordinate plane consider the set S of all points with integer coordinates. For a positive integer k, two distinct points A, $B \in S$ will be called k-*friends* if there is a point $C \in S$ such that the area of the triangle ABC is equal to k. A set $T \subset S$ will be called k-*clique* if every two points in T are k-friends. Find the least positive integer k for which there exits a k-clique with more than 200 elements.

Proposed by Jorge Tipe, Peru

2 For every $n \in \mathbb{N}$ let $d(n)$ denote the number of (positive) divisors of n. Find all functions $f : \mathbb{N} \to \mathbb{N}$ with the following properties: - $d\left(f(x)\right) = x$ for all $x \in \mathbb{N}$.
- $f(xy)$ divides $(x-1)y^{xy-1}f(x)$ for all x, $y \in \mathbb{N}$.

Proposed by Bruno Le Floch, France

3 Let A, B, C, M points in the plane and no three of them are on a line. And let A', B', C' points such that $MAC'B, MBA'C$ and $MCB'A$ are parallelograms:

(a) Show that

$$\overline{MA} + \overline{MB} + \overline{MC} < \overline{AA'} + \overline{BB'} + \overline{CC'}.$$

(b) Assume segments AA', BB' and CC' have the same length. Show that $2\left(\overline{MA} + \overline{MB} + \overline{MC}\right) \leq \overline{AA'} + \overline{BB'} + \overline{CC'}$. When do we have equality?

– TST 7

1 For which $n \geq 2, n \in \mathbb{N}$ are there positive integers $A_1, A_2, \ldots, A_n$ which are not the same pairwise and have the property that the product $\prod_{i=1}^{n}(A_i + k)$ is a power for each natural number k.

2 In Skinien there 2009 towns where each of them is connected with exactly 1004 other town by a highway. Prove that starting in an arbitrary town one can make a round trip along the highways such that each town is passed exactly once and finally one returns to its starting point.

3 There is given a convex quadrilateral $ABCD$. Prove that there exists a point P inside the quadrilateral such that

$$\angle PAB + \angle PDC = \angle PBC + \angle PAD = \angle PCD + \angle PBA = \angle PDA + \angle PCB = 90^\circ$$

if and only if the diagonals AC and BD are perpendicular.

Proposed by Dusan Djukic, Serbia

– Pre-TST 1

1 Let $p > 7$ be a prime which leaves residue 1 when divided by 6. Let $m = 2^p - 1,$ then prove $2^{m-1} - 1$ can be divided by $127m$ without residue.

2 Let triangle ABC be perpendicular at $A.$ Let M be the midpoint of segment $\overline{BC}.$ Point D lies on side $\overline{AC}$ and satisfies $|AD| = |AM|.$ Let $P \neq C$ be the intersection of the circumcircle of triangles AMC and $BDC.$ Prove that CP bisects the angle at C of triangle $ABC.$

3 Initially, on a board there a positive integer. If board contains the number $x,$ then we may additionally write the numbers $2x+1$ and $\frac{x}{x+2}.$ At some point 2008 is written on the board. Prove, that this number was there from the beginning.

– Pre-TST 2

1 Let n be a positive integer and let p be a prime number. Prove that if a, b, c are integers (not necessarily positive) satisfying the equations

$$a^n + pb = b^n + pc = c^n + pa$$

then $a = b = c$.

Proposed by Angelo Di Pasquale, Australia

2 Let $ABCD$ be a convex quadrilateral and let P and Q be points in $ABCD$ such that $PQDA$ and $QPBC$ are cyclic quadrilaterals. Suppose that there exists a point E on the line segment PQ such that $\angle PAE = \angle QDE$ and $\angle PBE = \angle QCE$. Show that the quadrilateral $ABCD$ is cyclic.

Proposed by John Cuya, Peru

3 The 16 fields of a 4×4 checker board can be arranged in 18 lines as follows: the four lines, the four columns, the five diagonals from north west to south east and the five diagonals from north east to south west. These diagonals consists of 2,3 or 4 edge-adjacent fields of same colour; the corner fields of the chess board alone do not form a diagonal. Now, we put a token in 10 of the 16 fields. Each of the 18 lines contains an even number of tokens contains a point. What is the highest possible point number when can be achieved by optimal placing of the 10 tokens. Explain your answer.