Germany Team Selection Test 2010

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by orl, April

– VAIMO 1

1 The quadrilateral $ABCD$ is a rhombus with acute angle at A. Points M and N are on segments $\overline{AC}$ and $\overline{BC}$ such that $|DM| = |MN|$. Let P be the intersection of AC and DN and let R be the intersection of AB and DM. Prove that $|RP| = |PD|$.

2 Prove or disprove that $\forall a, b, c, d \in \mathbb{R}^+$ we have the following inequality:

$$3 \leq \frac{4a+b}{a+4b} + \frac{4b+c}{b+4c} + \frac{4c+a}{c+4a} < \frac{33}{4}$$

3 Determine all $(m, n) \in \mathbb{Z}^+ \times \mathbb{Z}^+$ which satisfy $3^m - 7^n = 2$.

– VAIMO 2

1 Consider 2009 cards which are lying in sequence on a table. Initially, all cards have their top face white and bottom face black. The cards are enumerated from 1 to 2009. Two players, Amir and Ercole, make alternating moves, with Amir starting. Each move consists of a player choosing a card with the number k such that $k < 1969$ whose top face is white, and then this player turns all cards at positions $k, k+1, \ldots, k+40$. The last player who can make a legal move wins.

(a) Does the game necessarily end?
(b) Does there exist a winning strategy for the starting player?

Also compare shortlist 2009, combinatorics problem C1.

2 Given a cyclic quadrilateral $ABCD$, let the diagonals AC and BD meet at E and the lines AD and BC meet at F. The midpoints of AB and CD are G and H, respectively. Show that EF is tangent at E to the circle through the points E, G and H.

Proposed by David Monk, United Kingdom

3 A positive integer N is called *balanced*, if $N = 1$ or if N can be written as a product of an even number of not necessarily distinct primes. Given positive integers a and b, consider the polynomial P defined by $P(x) = (x+a)(x+b)$.
(a) Prove that there exist distinct positive integers a and b such that all the number $P(1)$, $P(2)$,. . ., $P(50)$ are balanced.
(b) Prove that if $P(n)$ is balanced for all positive integers n, then $a = b$.

Proposed by Jorge Tipe, Peru

– AIMO 1

1 Find the largest possible integer k, such that the following statement is true:
Let 2009 arbitrary non-degenerated triangles be given. In every triangle the three sides are coloured, such that one is blue, one is red and one is white. Now, for every colour separately, let us sort the lengths of the sides. We obtain

$$\begin{array}{rll} & b_1 \leq b_2 \leq \ldots \leq b_{2009} & \text{the lengths of the blue sides} \\ & r_1 \leq r_2 \leq \ldots \leq r_{2009} & \text{the lengths of the red sides} \\ \text{and} & w_1 \leq w_2 \leq \ldots \leq w_{2009} & \text{the lengths of the white sides} \end{array}$$

Then there exist k indices j such that we can form a non-degenerated triangle with side lengths b_j, r_j, w_j.

Proposed by Michal Rolinek, Czech Republic

2 Let P be a polygon that is convex and symmetric to some point O. Prove that for some parallelogram R satisfying $P \subset R$ we have

$$\frac{|R|}{|P|} \leq \sqrt{2}$$

where $|R|$ and $|P|$ denote the area of the sets R and P, respectively.

Proposed by Witold Szczechla, Poland

3 On a 999×999 board a *limp rook* can move in the following way: From any square it can move to any of its adjacent squares, i.e. a square having a common side with it, and every move must be a turn, i.e. the directions of any two consecutive moves must be perpendicular. A *non-intersecting route* of the limp rook consists of a sequence of pairwise different squares that the limp rook can visit in that order by an admissible sequence of moves. Such a non-intersecting route is called *cyclic*, if the limp rook can, after reaching the last square of the route, move directly to the first square of the route and start over.
How many squares does the longest possible cyclic, non-intersecting route of a limp rook visit?

Proposed by Nikolay Beluhov, Bulgaria

– AIMO 2

1 Let f be a non-constant function from the set of positive integers into the set of positive integer, such that $a-b$ divides $f(a)-f(b)$ for all distinct positive integers a, b. Prove that there exist infinitely many primes p such that p divides $f(c)$ for some positive integer c.

Proposed by Juhan Aru, Estonia

2 Five identical empty buckets of 2-liter capacity stand at the vertices of a regular pentagon. Cinderella and her wicked Stepmother go through a sequence of rounds: At the beginning of every round, the Stepmother takes one liter of water from the nearby river and distributes it arbitrarily over the five buckets. Then Cinderella chooses a pair of neighbouring buckets, empties them to the river and puts them back. Then the next round begins. The Stepmother goal's is to make one of these buckets overflow. Cinderella's goal is to prevent this. Can the wicked Stepmother enforce a bucket overflow?

Proposed by Gerhard Woeginger, Netherlands

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3 Let $ABCD$ be a circumscribed quadrilateral. Let g be a line through A which meets the segment BC in M and the line CD in N. Denote by I_1, I_2 and I_3 the incenters of $\triangle ABM$, $\triangle MNC$ and $\triangle NDA$, respectively. Prove that the orthocenter of $\triangle I_1I_2I_3$ lies on g.

Proposed by Nikolay Beluhov, Bulgaria

– AIMO 3

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1 A sequence (a_n) with $a_1 = 1$ satisfies the following recursion: In the decimal expansion of a_n (without trailing zeros) let k be the smallest digest then $a_{n+1} = a_n + 2^k$. How many digits does $a_{9\cdot 10^{2010}}$ have in the decimal expansion?

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2 Let ABC be a triangle with incenter I and let X, Y and Z be the incenters of the triangles BIC, CIA and AIB, respectively. Let the triangle XYZ be equilateral. Prove that ABC is equilateral too.

Proposed by Mirsaleh Bahavarnia, Iran

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3 Let $P(x)$ be a non-constant polynomial with integer coefficients. Prove that there is no function T from the set of integers into the set of integers such that the number of integers x with $T^n(x) = x$ is equal to $P(n)$ for every $n \geq 1$, where T^n denotes the n-fold application of T.

Proposed by Jozsef Pelikan, Hungary

– AIMO 4

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1 Let ABC be a triangle. The incircle of ABC touches the sides AB and AC at the points Z and Y, respectively. Let G be the point where the lines BY and CZ meet, and let R and S be points such that the two quadrilaterals $BCYR$ and $BCSZ$ are parallelogram.
Prove that $GR = GS$.

Proposed by Hossein Karke Abadi, Iran

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2 For an integer $m \geq 1$, we consider partitions of a $2^m \times 2^m$ chessboard into rectangles consisting of cells of chessboard, in which each of the 2^m cells along one diagonal forms a separate rectangle of side length 1. Determine the smallest possible sum of rectangle perimeters in such a partition.

Proposed by Gerhard Woeginger, Netherlands

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3 Find all functions f from the set of real numbers into the set of real numbers which satisfy for all x, y the identity

$$f\left(xf(x+y)\right) = f\left(yf(x)\right) + x^2$$

Proposed by Japan

– AIMO 5

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1 In the plane we have points P, Q, A, B, C such triangles APQ, QBP and PQC are similar accordantly (same direction). Then let A' (B', C' respectively) be the intersection of lines BP and CQ (CP and AQ; AP and BQ, respectively.) Show that the points A, B, C, A', B', C' lie on a circle.

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2 We are given $m, n \in \mathbb{Z}^+$. Show the number of solution $4-$tuples (a, b, c, d) of the system

$$\begin{aligned} ab + bc + cd - (ca + ad + db) &= m \\ 2\left(a^2 + b^2 + c^2 + d^2\right) - (ab + ac + ad + bc + bd + cd) &= n \end{aligned}$$

is divisible by 10.

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3 Find all functions $f : \mathbb{R} \to \mathbb{R}$ such that

$$f(x)f(y) = (x+y+1)^2 \cdot f\left(\frac{xy-1}{x+y+1}\right)$$

$\forall x, y \in \mathbb{R}$ with $x + y + 1 \neq 0$ and $f(x) > 1\ \forall x > 0.$

– AIMO 6

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1 Let $a \in \mathbb{R}$. Show that for $n \geq 2$ every non-real root z of polynomial $X^{n+1} - X^2 + aX + 1$ satisfies the condition $|z| > \frac{1}{\sqrt[n]{n}}$.

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2 Determine all $n \in \mathbb{Z}^+$ such that a regular hexagon (i.e. all sides equal length, all interior angles same size) can be partitioned in finitely many $n-$gons such that they can be composed into n congruent regular hexagons in a non-overlapping way upon certain rotations and translations.

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3 Let f be any function that maps the set of real numbers into the set of real numbers. Prove that there exist real numbers x and y such that

$$f\left(x-f(y)\right)>yf(x)+x$$

Proposed by Igor Voronovich, Belarus

– AIMO 7

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1 For any integer $n\geq 2$, let $N(n)$ be the maxima number of triples (a_i, b_i, c_i), $i=1,\ldots,N(n)$, consisting of nonnegative integers a_i, b_i and c_i such that the following two conditions are satisfied:
- $a_i+b_i+c_i=n$ for all $i=1,\ldots,N(n)$,
- If $i\neq j$ then $a_i\neq a_j$, $b_i\neq b_j$ and $c_i\neq c_j$
Determine $N(n)$ for all $n\geq 2$.

Proposed by Dan Schwarz, Romania

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2 Let a, b, c be positive real numbers such that $ab+bc+ca\leq 3abc$. Prove that

$$\sqrt{\frac{a^2+b^2}{a+b}}+\sqrt{\frac{b^2+c^2}{b+c}}+\sqrt{\frac{c^2+a^2}{c+a}}+3\leq \sqrt{2}\left(\sqrt{a+b}+\sqrt{b+c}+\sqrt{c+a}\right)$$

Proposed by Dzianis Pirshtuk, Belarus

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3 Find all positive integers n such that there exists a sequence of positive integers a_1, a_2,. . ., a_n satisfying:

$$a_{k+1}=\frac{a_k^2+1}{a_{k-1}+1}-1$$

for every k with $2\leq k\leq n-1$.

Proposed by North Korea
