Germany Team Selection Test 2014

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– VAIMO 1

1 In Sikinia we only pay with coins that have a value of either 11 or 12 Kulotnik. In a burglary in one of Sikinia's banks, 11 bandits cracked the safe and could get away with 5940 Kulotnik. They tried to split up the money equally - so that everyone gets the same amount - but it just doesn't worked. After a while their leader claimed that it actually isn't possible.
Prove that they didn't get any coin with the value 12 Kulotnik.

2 Let $ABCD$ be a convex cyclic quadrilateral with $AD=BD$. The diagonals AC and BD intersect in E. Let the incenter of triangle $\triangle BCE$ be I. The circumcircle of triangle $\triangle BIE$ intersects side AE in N.
Prove

$$AN \cdot NC = CD \cdot BN.$$

3 Let $a_1 \leq a_2 \leq \cdots$ be a non-decreasing sequence of positive integers. A positive integer n is called *good* if there is an index i such that $n=\dfrac{i}{a_i}$.
Prove that if 2013 is *good*, then so is 20.

– VAIMO 2

1 Let n be an positive integer. Find the smallest integer k with the following property; Given any real numbers $a_1, \cdots, a_d$ such that $a_1+a_2+\cdots+a_d=n$ and $0 \leq a_i \leq 1$ for $i=1,2,\cdots,d$, it is possible to partition these numbers into k groups (some of which may be empty) such that the sum of the numbers in each group is at most 1.

2 Let $\mathbb{Z}_{>0}$ be the set of positive integers. Find all functions $f: \mathbb{Z}_{>0} \rightarrow \mathbb{Z}_{>0}$ such that

$$m^2 + f(n) \mid mf(m) + n$$

for all positive integers m and n.

3 In a triangle ABC, let D and E be the feet of the angle bisectors of angles A and B, respectively. A rhombus is inscribed into the quadrilateral $AEDB$ (all vertices of the rhombus lie on different sides of $AEDB$). Let φ be the non-obtuse angle of the rhombus. Prove that $\varphi \leq \max\{\angle BAC, \angle ABC\}$.