Mathematical Olympiad Finals 1995

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1 Let $n \geq 2$ be integers and r be positive integers such that r is not the multiple of n, and let g be the greatest common measure of n and r. Prove that

$$\sum_{i=1}^{n-1} \left\{\frac{ri}{n}\right\} = \frac{1}{2}(n-g).$$

where $\{x\}$ is the fractional part, that is to say, which means the value by subtracting x from the maximum integer value which is equal or less than x.

2 Find all the non-constant rational function of $f(x)$ and real numbers a satisfying $\{f(x)\}^2 - a = f(x^2)$.
Here a rational function of x is the equation expressed by the ratio of two polynominals of x.

3 Given a convex pentagon $ABCDE$. Let S, R be the intersection points of AC or AD and BE respectively, and let the intersection points T, P of CA or CE and BD respectively.Let Q be the intersection point of CE and AD. If all of $\triangle ASR$, $\triangle BTS$, $\triangle CPT$, $\triangle DQP$, and $\triangle ERQ$ have the area of $1,$ then find the area of the following pentagons.

(1) The pentagon $PQRST$.
(2) The pentagon $ABCDE$.

4 The sequence $\{a_1,\ a_2,\ \cdots\}$ is defined by $a_{2n} = a_n,\ a_{2n+1} = (-1)^n$. A point P moves on the coordinate plane as follows.

(1) Let P_0 be the origin, P moves in a distance of 1 from P_0 toward in the positive direction of x-axis, Denote this point by P_i.

(2) After P has moved to P_i, it turns 90° to the left and moves in a distance of 1 when $a_i = 1$, and turns 90° to the right and moves in a distance of 1 when $a_i = -1$. Denote this point by P_{i+1},where $i = 1, 2, \cdots$. Prove that P can't pass on the same segment more than two times.

5 Let k, n be integers such that $1 \leq k \leq n$,and let $a_1,\ a_2,\ \cdots,\ a_k$ be numbers satisfying the following equations.

$$\begin{cases} a_1 + \cdots\cdots + a_k = n \\ a_1^2 + \cdots\cdots + a_k^2 = n \\ \qquad\quad \vdots \\ a_1^k + \cdots\cdots + a_k^k = n \end{cases}$$

Prove that

$$(x+a_1)(x+a_2)\cdots(x+a_k) = x^k + {}_nC_1\, x^{k-1} + {}_nC_2\, x^{k-2} + \cdots + {}_nC_k.$$

where ${}_iC_j$ is a binomial coefficient which means $\frac{i\cdot(i-1)\cdots(i-j+1)}{j\cdot(j-1)\cdots 2\cdot 1}$.