Mathematical Olympiad Finals 1996

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by WakeUp

1 A plane is partitioned into triangles. Let $\mathcal{T}_0$ denote the set of vertices of triangles in the partition. Let ABC be a triangle with $A, B, C \in \mathcal{T}_0$ and θ be its smallest angle. Assume that no point of $\mathcal{T}_0$ lies inside the circumcircle of $\triangle ABC$. Prove that there exists a triangle σ in the partition such that its intersection with $\triangle ABC$ is nonempty and whose every angle is greater than θ.

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2 Let m, n be positive integers with $(m, n) = 1$. Find $(5^m + 7^m, 5^n + 7^n)$.

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3 Let $x > 1$ be a real number which is not an integer. For each $n \in \mathbb{N}$, let $a_n = \lfloor x^{n+1} \rfloor - x\lfloor x^n \rfloor$. Prove that the sequence (a_n) is not periodic.

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4 Let θ be the maximum of the six angles between six edges of a regular tetrahedron in space and a fixed plane. When the tetrahedron is rotated in space, find the maximum of θ.

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5 Let q be a real number with $\frac{1+\sqrt{5}}{2} < q < 2$. If a positive integer n is represented in binary system as $2^k + a_{k-1}2^{k-1} + \cdots + 2a_1 + a_0$, where $a_i \in \{0, 1\}$, define

$$p_n = q^k + a_{k-1}q^{k-1} + \cdots + qa_1 + a_0.$$

Prove that there exist infinitely many positive integers k with the property that there is no $l \in \mathbb{N}$ such that $p_{2k} < p_l < p_{2k+1}$.
