Mathematical Olympiad Finals 1998

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1 Let $p \geq 3$ be a prime, and let p points $A_0, \ldots, A_{p-1}$ lie on a circle in that order. Above the point $A_{1+\cdots+k-1}$ we write the number k for $k = 1, \ldots, p$ (so 1 is written above A_0). How many points have at least one number written above them?

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2 A country has 1998 airports connected by some direct flights. For any three airports, some two are not connected by a direct flight. What is the maximum number of direct flights that can be offered?

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3 Let $P_1, \ldots P_n$ be the sequence of vertices of a closed polygons whose sides may properly intersect each other at points other than the vertices. The external angle at P_i is defined as 180° minus the angle of rotation about P_i required to bring the ray P_iP_{i-1} onto the ray P_iP_{i+1}, taken in the range $(0^\circ, 360^\circ)$. (Here $P_0 = P_n$ and $P_1 = P_{n+1}$). Prove that if the sum of the external angles is a multiple of 720°, then the number of self-intersections is odd.

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4 Let $c_{n,m}$ be the number of permutations of $\{1, \ldots, n\}$ which can be written as the product of m transpositions of the form $(i, i+1)$ for some $i = 1, \ldots, n-1$ but not of $m-1$ suct transpositions. Prove that for all $n \in \mathbb{N}$,

$$\sum_{m=0}^{\infty} c_{n,m} t^m = \prod_{i=1}^{n} (1 + t + \cdots + t^{i-1}).$$

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5 On each of 12 points around a circle we place a disk with one white side and one black side. We may perform the following move: select a black disk, and reverse its two neighbors. Find all initial configurations from which some sequence of such moves leads to the position where all disks but one are white.
